Intersections of random walks

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SUMMARY

We study the large deviation behaviour of simple random walks in dimension three or more in this thesis. The first part of the thesis concerns the number of lattice sites visited by the random walk. We call this the range of the random walk. We derive a large deviation principle for the probability that the range of simple random walk deviates from its mean. Our result describes the behaviour for deviation below the typical value. This is a result analogous to that obtained by van den Berg, Bolthausen, and den Hollander for the volume of the Wiener sausage.

In the second part of the thesis, we are interested in the number of lattice sites visited by two independent simple random walks starting at the origin. We call this the intersection of ranges. We derive a large deviation principle for the probability that the intersection of ranges by time n exceeds a multiple of n. This is also an analogous result of the intersection volume of two independent Wiener sausages.

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Contents

Chapter 1

Introduction

The random walk model had been introduced and well-studied since the early 19th century, and still remains one of the most fundamental models of discrete time stochastic processes. There are a lot of applications for random walk models in many areas such as in biology, ecology, physics and computer science, as well as providing models in gambling theory, stock price and potential theory. The recurrence property had been studied by Polya [Pól21] and he showed that the probability that the random walk will *not* return to the origin is zero for dimension one and two, i.e. random walk is guaranteed to return to the origin. However, this non-return probability is non-zero for dimension three or more. Later on, Dvoretzky and Erdös [DE51] investigated the problem on the range of random walk, and they showed that the expected number of lattice sites visited by a random walk depends on the dimension, and is also associated with the non-return probability. Then, Donsker and Varadhan [DV79] derived the limit behaviour of the range of a random walk in an exponent form. This celebrated result leads to the study of the large deviation behaviour of the range of random walk.

More recently, there is also development on the study of the intersection of multiple independent random walks. We concentrate on the intersections of the ranges, i.e. we count how many sites have been visited by two or more independent random walks. The intersection properties of random walks have been studied extensively in the past fifteen years. The notable result by Dvoretzky, Erdös and Kakutani [DEK50] shows that the number of intersection sites for two independent random walks is infinite in dimension four or less, and finite in dimension five or more. This leads to the study of the infinite-time intersection behaviour.

In the first part of this thesis, we will concentrate on the study of the range of a single random walk on $\mathbb{Z}^d$. We compute the rate of decay of the probability of the event that the random walk makes a visit on the lattice site *less than usual* on the mean scale. Our study shows that this event satisfies a large deviation principle, which means that the probability of this event decreases exponentially. This result agrees with the analogous result in the continuous case studied by van den Berg, Bolthausen and den Hollander [BBH01]. This part of the thesis is based on the material in the submitted paper

> [Phe11] P. PHETPRADAP. Large deviations for the range of a simple random walk, *ESAIM*, submitted, 2011.

The second part of this thesis will focus on the problem of the intersection of the range of independent random walks. We consider the case of two independent random walks and find the probability that the number of intersection made by the random walks is larger than usual. This result agrees with the analogous result in the continuous case studied by van den Berg et al. [BBH04]. The long term aim of this problem is to study the infinite-time intersection behaviour, as we know that the number of intersections is finite for dimension five or more.

The remainder of this thesis is structured as follows: In this chapter, we will first introduce the main intersection quantities in Section 1.1. Then, we will describe the development of the main quantities of interest and show our result. This will be done separately on the range of random walk and the intersection of independent ranges in, respectively, Section 1.2 and Section 1.3. Chapter 2 contains the proof for the result on the range, while Chapter 3 gives the proofs for the result on the intersection of ranges. Finally, we summarise the technique we have learnt and discuss an open problem in Chapter 4.

1.1 Random walk model and quantities of interest

In this section, we provide the general set up of $d-$dimensional *random walk* model and point out our main quantities of interest for the thesis. We also provide the set up of a *Wiener sausage*.

Let $(S_n : n = 1, 2, \ldots)$ be a simple random walk on the integer lattice $\mathbb{Z}^d$, i.e.

$$S_n = X_1 + X_2 + \ldots + X_n \qquad n = 1, 2, \ldots$$

and $X_1, X_2, \ldots$ a sequence of independent, identically distributed random vectors with $\mathbb{P}(X_1 = e_i) = \mathbb{P}(X_1 = -e_i) = \frac{1}{2d}$, where $e_1, e_2, \ldots, e_d$ are the orthogonal unit vectors on $\mathbb{Z}^d$. Unless stated otherwise, we assume that $S_0 = 0$. We *always* consider (S_n) as a simple random walk throughout the thesis. Also, we let $(S_n^1), (S_n^2), \ldots$ be independent copies of (S_n).
In this thesis, we will pay particular attention to these two quantities:

- **Range of the random walk**: The number of distinct sites on $\mathbb{Z}^d$ visited by a random walk up to time n,

$$R_n = \sharp\{S_1, \ldots, S_n\}. \tag{1.1.1}$$

- **Intersection of the independent ranges**: The number of distinct sites on $\mathbb{Z}^d$ visited by all k independent random walks up to time n,

$$J_n^k = \sharp\big\{\{S_j^1\}_{1\leq j\leq n} \cap \{S_j^2\}_{1\leq j\leq n} \cap \ldots \cap \{S_j^k\}_{1\leq j\leq n}\big\}. \tag{1.1.2}$$

  In this thesis, we will mainly focus in the case $k = 2$. Later on, we will write:

$$J_n := J_n^2. \tag{1.1.3}$$

We may notice that there is a relation between R_n and J_n^k. Note that we can write R_n as:

$$R_n = \sum_{x\in\mathbb{Z}^d} \mathbf{1}\{S_i = x \text{ for some } 1 \leq i \leq n\}, \tag{1.1.4}$$

where $\mathbf{1}\{\cdot\}$ is an indicator function. While for J_n^k we have

$$\begin{aligned} J_n^k &= \sum_{x\in\mathbb{Z}^d} \mathbf{1}\{S_{i_1}^1 = S_{i_2}^2 = \ldots = S_{i_k}^k = x \text{ for some } 1 \le i_1, i_2, \ldots, i_k \le n\} \\ &= \sum_{x\in\mathbb{Z}^d} \prod_{l=1}^{k} \mathbf{1}\{S_i^l = x \text{ for some } 1 \le i \le n\}. \end{aligned} \tag{1.1.5}$$

Our main aim for this thesis is to study the large deviation behaviour of these quantities. There are a number of results that have been developed in the large deviation sense, in particular by Donsker and Varadhan [DV79], Hamana and Kesten [HK01] and recent book by Chen [Che10]. The results and heuristic arguments of the proof will be shown in Section1.2.1 for R_n and Section 1.3.1 for J_n.

Let $\beta(t), t \ge 0$ be a standard Brownian motion in $\mathbb{R}^d$ starting at the origin. We also denote $\beta_1(t), \beta_2(t), \ldots$ as independent copies of $\beta(t)$. Define $W^a(t)$ to be a *Wiener sausage* up to time t with radius $a > 0$ by

$$W^a(t) := \bigcup_{0\le s\le t} B_a\big(\beta(s)\big), \qquad t \ge 0, \tag{1.1.6}$$

where $B_a(x)$ is the open ball with radius a around $x \in \mathbb{R}^d$. Similarly, define $W_1^a(t), \ldots, W_k^a(t)$ as the Wiener sausages associated with $\beta_1(t), \ldots, \beta_k(t)$. The Wiener sausage is one of the simplest examples of a non-Markovian functional of Brownian motion. We also define $V_k^a(t)$ to be the intersection set of all $W_1^a(t), \ldots, W_k^a(t)$, i.e.

$$V_k^a(t) := \bigcap_{i=1}^{k} W_i^a(t), \tag{1.1.7}$$

with our usual simplification $V^a(t) := V_2^a(t)$. We also let $|W^a(t)|$ to be the *volume* of $W^a(t)$, and $|V_k^a(t)|$ to be the *intersection volume* up to time t of $W_1^a(t), \ldots, W_k^a(t)$. Note that we can also represent $W^a(t)$ and $V_k^a(t)$ in a similar to the form of (1.1.4) and (1.1.5).

It is well known that Brownian is the scaling limit of random walk. Therefore, we would expect similar behaviour in limits for random walk and Brownian motion. It turns out that problems on R_n are analogous results of $|W^a(t)|$, while problems on J_n^k are coupled with $|V_k^a(t)|$.

There is also a development in the study of limiting behaviour of $|W^a(t)|$ and $|V_k^a(t)|$ in the large deviation sense, notably by Donsker and Varadhan [DV75], Bolthausen [Bol90], Hamana and Kesten [HK01] and van den Berg, Bolthausen and den Hollander [BBH01, BBH04]. We also include the results and optimal strategies of the proofs in Section

1.2.1 for $|W^a(t)|$ and Section 1.3.1 for $|V^a(t)|$.

1.2 The ranges

In this section we will focus on problems and developments on R_n and $|W^a(t)|$. Firstly, we will give general overviews and known results in the classical case and the large deviation case in Section 1.2.1. Then, we will present our main result and give extra comments in Section 1.2.2. Finally, the outline of the proof of our results will be explained in Section 1.2.3.

1.2.1 Overview

We start this section with the obvious fact that R_n is bounded due to discreteness property the random walk, i.e.

$$0 < R_n \leq n.$$

Remarks:

1. The random walk conditional on the event $R_n = n$ is called *self-avoiding walk* which was introduced as a polymer model in Physics and has been popularly studied since. The recent question is to find the existence and conformal invariance of the scaling limit of self-avoiding walk which is conjectured to be described by Schramm-Loewner evolution. However, we will not mention self-avoiding walk in this thesis. Readers can find further reading material at [Law96] for example.

2. There are also bounds for R_n in an exponent form, for example:

$$\sup_{n \geq 1} \mathbb{E} \exp \left(\frac{\theta}{n^{2/3}} (R_n - \mathbb{E}R_n) \right) < \infty \qquad \forall \theta > 0. \tag{1.2.1}$$

   The full proof can be seen in Theorem 6.3.2 of [Che10].

One of the typical questions to be asked is what is the expected value and the variance of R_n? This question has been answered by Dvoretzky and Erdös [DE51]. Before we quote the theorem, we first need to define the non-return probability of random walk on $\mathbb{Z}^d$:

$$\kappa := \kappa(d) = \mathbb{P}(S_i \neq 0 \text{ for all } i \geq 1), \tag{1.2.2}$$

i.e. κ is the probability of the event that the random will never return to the origin. The value of κ depends on the dimension d. By the recurrence property of random walk (e.g., by Polya [Pól21]), it can be deduced that the non-return probability is zero for dimension one and two and positive for dimension three or more.

Theorem 1.2.1. *As $n \to \infty$,*

$$\mathbb{E}(R_n) = \begin{cases} \kappa n + O(n^{1/2}), & \text{if } d = 3, \\ \kappa n + O(\log n), & \text{if } d = 4, \\ \kappa n + c_d + O(n^{2-d/2}), & \text{if } d \geq 5, \end{cases}$$

where c_d are positive constants depending on the dimension $d \geq 5$, and

$$Var(R_n) \leq \begin{cases} O(n^{3/2}), & \text{if } d = 3, \\ O(n \log n), & \text{if } d = 4, \\ O(n), & \text{if } d \geq 5. \end{cases}$$

Furthermore, it also satisfies the strong law of large numbers,

$$\lim_{n\to\infty} \frac{R_n}{\mathbb{E}R_n} = 1 \text{ almost surely.}$$

Proof(sketch). To work out the expected value, define an indicator function

$$\gamma_k = \begin{cases} 1, & \text{if } S_k \neq S_i \text{ for } i = 1, \ldots, k-1, \\ 0, & \text{otherwise }. \end{cases}$$

Now, we can see that

$$\begin{aligned} \mathbb{E}R_n = \mathbb{E}\sum_{k=1}^{n} \gamma_k &= \sum_{k=1}^{n} \mathbb{P}(S_k \neq S_i \text{ for } i = 1, \ldots, k-1) \\ &= \sum_{k=1}^{n} \mathbb{P}\Big(\sum_{j=i+1}^{k} X_j \neq 0 \text{ for } i = 1, \ldots, k-1\Big) \\ &= \sum_{k=1}^{n} \mathbb{P}\Big(\sum_{j=1}^{k-i} X_j \neq 0 \text{ for } i = 1, \ldots, k-1\Big) \\ &= \sum_{k=1}^{n} \mathbb{P}(S_i \neq 0 \text{ for } i = 1, \ldots, k-1). \end{aligned} \tag{1.2.3}$$

However, we can see that

$$\lim_{k\to\infty} \mathbb{P}(S_i \neq 0 \text{ for } i = 1, \ldots, k-1) = \kappa,$$

and hence

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbb{P}(S_i \neq 0 \text{ for } i = 1, \ldots, k-1) = \kappa.$$

The error term depends on the dimension d. The proof for the upper bound of $\mathrm{Var}(R_n)$ is omitted.

$\square$

The non-return probability κ will later play a key role in the analysis of the problem on the range. Also it is worth to comment that the expected value of R_n is of order n. Note that, Theorem 1.2.1 only gives upper bounds for $\mathrm{Var}(R_n)$. This was later developed to give the exact order of the variance by Jain and Orey [JO68] for $d \geq 5$ and by Jain and Pruitt [JP71] for $d \geq 3$:

$$\mathrm{Var}(R_n) \asymp \begin{cases} O(n \log n), & \text{for } d = 3, \\ O(n), & \text{for } d \geq 4, \end{cases} \tag{1.2.4}$$

where we write $f(n) \asymp O\big(g(n)\big)$ implies that, for some positive constant C_1, C_2,

$$C_1 g(n) \leq f(n) \leq C_2 g(n). \tag{1.2.5}$$

Proof (sketch). We will give the sketch of the proof. Define:

$$Z_i^n = \mathbf{1}\{S_i \neq S_{i+1}, \ldots, S_i \neq S_n\}, \qquad 0 \leq i \leq n, Z_n^n = 1 \tag{1.2.6}$$

$$Z_i = \mathbf{1}\{S_i \neq S_{i+1}, S_i \neq S_{i+2}, \ldots\}, \qquad i \geq 0 \tag{1.2.7}$$

$$W_i^n = Z_i^n - Z_i. \tag{1.2.8}$$

i.e.

- Z_i^n is an indicator function of the event that after time i (where the position of the random walk is S_i), the random walk will not come back to lattice S_i by time n.
- Z_i is an indicator function of the event that after time i, the random walk will never come back to S_i.
- W_i^n is an indicator function of the event that after time i, the random walk will

not come back to S_i by time n, but it will eventually make a return to S_i after time n.

We can write R_n as

$$R_n = \sum_{i=1}^{n} Z_i^n = \sum_{i=1}^{n} Z_i + \sum_{i=1}^{n} W_i^n. \tag{1.2.9}$$

We will abbreviate the sums as:

$$Y_n := \sum_{i=1}^{n} Z_i, \qquad W_n := \sum_{i=1}^{n} W_i^n. \tag{1.2.10}$$

The idea to deduce $\mathrm{Var}(R_n)$ is to show that **(i)** For $d = 3$, $\mathrm{Var}(Y_n) = O(n \log n)$, **(ii)** For $d \geq 4$, $\mathrm{Var}(Y_n) = O(n)$, and **(iii)** $\mathrm{Var}(W_n) = o(\mathrm{Var}(Y_n))$.
Firstly, to show that $\mathrm{Var}(W_n) = o(\mathrm{Var}(Y_n))$, we calculate $\mathbb{E}W_i^n W_j^n$ for $i \leq j$. This was done in Lemma 4 of [JP71] and the formula of the expectation is explicitly given. Then, by expanding $\mathrm{Var}(\sum_{i=1}^n W_i^n)$ along with $\mathbb{E}W_i^n W_j^n$, we get

$$\mathbb{E}W_n^2 \asymp \begin{cases} O(n), & \text{for } d = 3, \\ O(\log^2 n), & \text{for } d = 4, \\ O(1), & \text{for } d = 5. \end{cases} \tag{1.2.11}$$

Now, to derive $\mathrm{Var}(Y_n)$, the key step is to deduce the covariances $\mathrm{cov}(Z_i, Z_j)$ for each $1 \leq i \leq j \leq n$. By Lemma 3 and Lemma 5 of [JP71], we get

$$a_j := \sum_{i=1}^{j-1} \mathrm{cov}(Z_i, Z_j) \asymp \begin{cases} O(\log j), & \text{for } d = 3, \\ O(1), & \text{for } d \geq 4. \end{cases} \tag{1.2.12}$$

Now, we deduce (1.2.4). For $d \geq 3$, we can see that

$$\mathrm{Var}(Z_i) = \mathbb{E}Z_i^2 - (\mathbb{E}Z_i)^2 = \kappa - \kappa^2 \tag{1.2.13}$$

Finally, we derive $\mathrm{Var}(Y_n)$. By (1.2.12) and (1.2.13) we get

$$\begin{aligned} \mathrm{Var}(Y_n) &= \sum_{i=1}^{n} \mathrm{Var}(Z_i) + 2 \sum_{j=1}^{n} \sum_{i=1}^{j-1} \mathrm{Cov}(Z_i, Z_j) \\ &\leq \big(\kappa(1-\kappa)\big) n + 2na_n \asymp \begin{cases} O(n \log n), & \text{for } d = 3, \\ O(n), & \text{for } d \geq 4. \end{cases} \end{aligned}$$

Note that, the result for $d \geq 4$ can be deduced since $\kappa(1-\kappa) + 2a_n$ is positive. Combine this result with (1.2.11), we can see that Var $(W_n) = o(\mathrm{Var}(Y_n))$. Hence, $\mathrm{Var}(R_n) \asymp$

$\mathrm{Var}(Y_n)$ by Schwarz inequality. $\square$

Now, the next question would be, how behaviours of R_n looks like? Would R_n satisfy the central limit theorem? The answer is "Yes, the central limit theorem also holds for R_n". This result was first proved in $d \geq 5$ by Jain and Orey [JO68] and later by Jain and Pruitt [JP71] for $d \geq 3$.

Theorem 1.2.2.

$$\frac{1}{\sqrt{n \log n}}(R_n - \mathbb{E}R_n) \xrightarrow{d} \mathcal{N}(0, \mathfrak{D}^2), \qquad d = 3,$$
$$\frac{1}{n}(R_n - \mathbb{E}R_n) \xrightarrow{d} \mathcal{N}(0, \tilde{\mathfrak{D}}^2), \qquad d \geq 4.$$

The exact forms of the variances $\mathfrak{D}^2, \tilde{\mathfrak{D}}^2$ can be found in [LGR91] as well as Theorem 5.5.3 of [Che10].

Proof (sketch). Consider $d \geq 4$. We will use similar notations as in the proof for the variance of R_n, in (1.2.6)-(1.2.9). Write:

$$R_n = \sum_{i=1}^{n} Z_i^n = \sum_{i=1}^{n} Z_i + \sum_{i=1}^{n} W_i^n.$$

We also abbreviate the sums as similar to (1.2.10):

$$Y_n := \sum_{i=1}^{n} Z_i, \qquad W_n := \sum_{i=1}^{n} W_i^n.$$

Next, it can be shown (e.g. by Chebychev's inequality) that

$$\mathbb{P}(W_n \geq \epsilon n^{1/2}) \overset{n\to\infty}{\longrightarrow} 0.$$

Hence $W_n/(\sigma\sqrt{n})$ converges to zero in probability. Therefore, to show that R_n has a Gaussian limit, we can only need to consider the sequence Y_n.

To do this, we will partition the random walk, each of timelength $m := \lfloor n^{1/3} \rfloor$ (while we ignore the continuity correction for the sketch proof). The key idea of the proof is to show that $(Y_n - \mathbb{E}Y_n)/\sigma\sqrt{n}$ satisfies Lindeberg's conditions. This will be done by the following: Define

$$U_i = \sum_{j=(i-1)m}^{im-1} Z_j^{im}, \qquad V_i = \sum_{j=(i-1)m}^{im-1} W_j^{im}.$$

Then, we write Y_n as:

$$Y_n - \kappa n = \sum_{k=0}^{m^2-1} (Z_k - \kappa) = \sum_{i=1}^{m^2}(U_i - \mathbb{E}U_i) - \sum_{i=1}^{m^2}(V_i - \mathbb{E}V_i).$$

Next, it can be checked that

$$\frac{\sum(V_i - \mathbb{E}V_i)}{\sigma\sqrt{n}} \xrightarrow{\mathbb{P}} 0,$$

therefore we only need to consider the sequence $U_i - \mathbb{E}U_i$. Then, it can be shown that Lindeberg's conditions are satisfied, i.e.:

$$\mathrm{Var}\Big(\sum_{i=1}^{m^2} \frac{U_i - \mathbb{E}U_i}{\sigma\sqrt{n}}\Big) = \frac{m^2}{\sigma^2 n}\mathrm{Var}R_m \sim 1$$
$$\sum_{i=1}^{m^2} \int_{|U_i - \mathbb{E}U_i| \geq \epsilon\sigma\sqrt{n}} \Big(\frac{U_i - \mathbb{E}U_i}{\sigma\sqrt{n}}\Big)^2 d\mathbb{P} \overset{n\to\infty}{\longrightarrow} 0,$$

where the second part of the condition come from the fact that $|U_i - \mathbb{E}U_i|/\sigma\sqrt{n} \leq m/\sigma\sqrt{n} \to 0$ as $n \to \infty$. Hence we can conclude that $(R_n - \mathbb{E}R_n)/\sigma\sqrt{n}$ is asymptotically normal.

In three dimensions, the problem is more delicate since the partition need to be of length $\lfloor n/\log n \rfloor$, and additional results are required to show that Lindeberg's conditions are satisfied. □

Remarks:

1. We also would like to mention analogous results in Wiener sausage case. We start with similar type of results on the classical behaviour. By Spitzer [Spi64] and Le Gall [LeG88]:

   $$\mathbb{E}|W^a(t)| \sim \kappa_a t, \qquad\qquad d \geq 3,$$

   where κ_a is the Newtonian capacity of $B_a(0)$, and

   $$\mathrm{Var}|W^a(t)| \sim \begin{cases} O(t\log t), & \text{for } d = 3, \\ O(t), & \text{for } d \geq 4. \end{cases}$$

   By comparison with Theorem 1.2.1 and (1.2.4), we can see that the expected mean and variance of R_n and $|W^a(t)|$ are on the same scale. Moreover, $|W^a(t)|$ satisfies the strong law of large numbers and the central limit theorem for $d \geq 3$.

2. Apart from problems on the range of random walk, there is also active study on

self-intersection local time problems of a random walk, which is defined by:

$$Q_n = \sum_{1 \le j \le k \le n} \mathbf{1}(S_j = S_k).$$

Self-intersections of random walks and Brownian motion have been studied intensively. They play a key role e.g. in description of polymer chains (Madras and Slade [MS93]).

Although we will not study the behaviours of Q_n in this thesis, there is a relation between R_n and Q_n. We can see that R_n and Q_n are related in negative ways, i.e. the more self-intersections, the smaller the range. However, it seems to be difficult to express the relation in mathematical formula. One surprise relation is that the behaviour of Q_n seems to be similar to R_n in high dimensions. For example,

$$\mathbb{E}Q_n = cn, \qquad\qquad \text{for } d \ge 3,$$

and

$$\mathrm{Var}(Q_n) \sim \begin{cases} O(n \log n), & \text{for } d = 3, \\ O(n), & \text{for } d \ge 4. \end{cases}$$

The central limit theorem is obtained by Chen [Che08], while the large deviation behaviour in high dimensions has been studied by Asselah [Ass08, Ass09]. See [Che10] for up-to-date results on the behaviours of the self-intersection local time.

Large deviation behaviour

We may categorise the large deviation-type problems on the range of random walk by the direction of events we considered. Therefore, there will be two main categories for this, namely downward direction (event of type $\{R_n \le f(n)\}$) and upward direction ($\{R_n \ge f(n)\}$).

1. Downward direction

The first result was from Donsker and Varadhan while they showed a limit behaviour in an exponent form. This was first done in the Wiener sausage case [DV75] and later in the random walk case [DV79].

Theorem 1.2.3. *Let $a > 0$. For any $\theta > 0$ and $d \geq 3$*

$$\lim_{n\to\infty} n^{-d/(d+2)} \log \mathbb{E} \exp(-\theta R_n) = -k(\theta, d), \tag{1.2.14}$$

$$\lim_{t\to\infty} t^{-d/(d+2)} \log \mathbb{E} \exp(-\theta |W^a(t)|) = -k(\theta, d), \tag{1.2.15}$$

where

$$k(\theta, d) = \theta^{2/(d+2)} \Big(\frac{d+2}{2}\Big)\Big(\frac{2\alpha_d}{d}\Big)^{d/(d+2)}$$

and α_d is the lowest eigenvalue of $-(1/2)\Delta$ for the sphere of unit volume in $\mathbb{R}^d$ with zero boundary values.

Remarks:

1. Obviously, both random walk and Brownian motion have the same limit. It is worth saying that the rate function does not depend on the radius of Wiener sausage a.

2. We can use the Gärtner-Ellis theorem to transform the result in the form of large deviation scale.

   Corollary 1.2.4. *For any $\nu > 0$,*

   $$\lim_{n\to\infty} n^{-d/(d+2)} \log \mathbb{P}(R_n \leq \nu n^{d/(d+2)}) = -J(\nu),$$

   where

   $$J(\nu) = \frac{1}{\nu}\Big(\frac{d+2}{2}\Big)\Big(\frac{d}{2\alpha_d}\Big)^{-\frac{2d}{d+2}} - \frac{1}{\nu^{d/2}}\Big(\frac{d}{2\alpha_d}\Big)^{\frac{d+2}{2}}.$$

   This implies that Theorem 1.2.3 gives a large deviation result in the downward direction for the scale $n^{d/(d+2)}$ which is *less* than the mean, which came as a surprising result.

   Proof (sketch). We rewrite (1.2.14) as:

   $$\lim_{m\to\infty} \frac{1}{m} \log \mathbb{E} \exp[m\nu X_m] = -k(\nu), \tag{1.2.16}$$

   where X_m is a family of random variable taking values in $\mathbb{R}$. By comparison with (1.2.14), we set $m = n^{d/(d+2)}$ and $X_m = -\frac{1}{m}R_n$. Since $k(\nu)$ is in explicit form and differentiable, we can apply the Gärtner-Ellis theorem. Hence, we get

   $$\lim_{m\to\infty} \frac{1}{m} \log \mathbb{P}(X_m \geq a) = -\sup_y \{ya + k(y)\}.$$

Now, by substitute the values of m and X_m, and set $b = -a$, we get

$$\lim_{n\to\infty} n^{-d/(d+2)} \log \mathbb{P}(R_n \leq bn^{d/(d+2)}) = -\sup_y \{-by + k(y)\}$$

Finally, we deduce the rate function $J(\nu)$ in Corollary 1.2.4. This can be done explicitly. □

3. For both cases, the optimal strategy to realise the large deviation is to stay inside a ball of size $n^{d/(d+2)}$ until time n and fill all the space within the ball entirely and nothing outside. The cost of this strategy leads to the exponential limit $k(\theta, d)$.

The next result in downward direction was done by van den Berg, Bolthausen and den Hollander [BBH01] and they show the result in the Wiener sausage case for large deviations on the scale of its mean.

Theorem 1.2.5. *Let $d \geq 3$ and $a > 0$. For every $b > 0$,*

$$\lim_{t\to\infty} \frac{1}{t^{\frac{d-2}{d}}} \log \mathbb{P}(|W^a(t)| \leq bt) = -I^{\kappa_a}(b), \tag{1.2.17}$$

where

$$I^{\kappa_a}(b) = \inf_{\phi \in \Phi^{\kappa_a}(b)} \Big[\frac{1}{2} \int_{\mathbb{R}^d} |\nabla \phi|^2(x)\, dx\Big], \tag{1.2.18}$$

with

$$\Phi^{\kappa_a}(b) = \big\{\phi \in H^1(\mathbb{R}^d)\colon \int_{\mathbb{R}^d} \phi^2(x) dx = 1, \int_{\mathbb{R}^d} (1 - e^{-\kappa_a \phi^2(x)}) dx \leq b\big\}. \tag{1.2.19}$$

Remarks:

1. Our main result is obtaining the analogous result for the similar problem in the random walk case. This will be explored in Section 1.2.2.

2. Note that Theorem 1.2.5 holds for every $b > 0$. However, for $b > \kappa_a$, the rate function in (1.2.18) is infinite.

3. We repeat the words used by the authors of [BBH01] to describe the optimal strategy of the problem

> The idea behind the theorem is that the optimal strategy for Brownian motion to realise the event $\{|W^a(t) \leq bt|\}$ is to behave like a Brownian motion in a drift field $xt^{1/d} \to (\nabla\phi/\phi)(x)$ for some smooth $\phi : \mathbb{R}^d \to [0, \infty)$. The cost of adopting this drift during a time t is

the exponential of $t^{(d-2)/d}$ times the integral in (1.2.18) to leading order. The effect of the drift is to push the Brownian motion towards the origin. Conditioned on adopting the drift, the Brownian motion spends time $\phi^2(x)$ per unit volume in the neighbourhood of $xt^{1/d}$, and it turns out that Wiener sausage covers a fraction $1 - \exp[-\kappa_a \phi^2(x)]$ of the space in that neighbourhood. The best choice of the drift field is therefore given by a minimiser of the variational problem in (1.2.18), or by a minimising sequence.

Therefore, the optimal strategy for the Wiener sausage is to cover only part of the space and to leave random holes whose size are of order 1 and whose density varies on scale $t^{1/d}$. Note that this strategy is more complicated than the strategy in Theorem 1.2.3. A large deviation *on the scale of the mean* does not squeeze all the empty space out of the Wiener sausage, and the limit in (1.2.17) depends on a.

2. Upward direction

The behaviour in the upward direction has been first studied in the Wiener sausage case. Van den Berg and Tóth [BT91] firstly develop the Donsker-Varadhan Wiener sausage result to show the asymptotic behaviour in exponent term of $\lambda|W^a(t)|$ where $\lambda > 0$. They showed that the limit

$$S(\lambda) := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E} \exp(\lambda |W^1(t)|)$$

is positive and finite in any dimension for any $\lambda > 0$. Moreover, for $d \geq 3$,

$$\max\{\omega_{d-1}^2 \lambda^2, d(d-2)\omega_d \lambda\} \leq S(\lambda) \leq 2^d d\omega_{d-1} e^{2^{d-1}\omega_{d-1}} \max\{\lambda, \lambda^2\},$$

where ω_d is the volume of the ball of radius 1 in $\mathbb{R}^d$. The result is later developed by Bolthausen and van den Berg [BB94] to show the limit results of $S(\lambda)$ when λ is either large or small. They showed that for $d \geq 3$

$$\lim_{\lambda\downarrow 0} \frac{S(\lambda)}{\lambda} = \kappa_a, \qquad \lim_{\lambda\to\infty} \frac{S(\lambda)}{\lambda^2} = (\omega_{d-1})^2,$$

where κ_a is the Newtonian capacity of the ball of radius a in $\mathbb{R}^d$. It is also mentioned that $\lambda \mapsto S(\lambda)$ is convex and lower semicontinuous.

The analogous result in the discrete case had been studied by Hamana [Ham01]. By setting

$$\Lambda_n(\theta) = \frac{1}{n} \log \mathbb{E}\big(e^{\theta R_n}\big), \tag{1.2.20}$$

then, note that for $\theta > 0$,

$$\mathbb{E}\big(e^{\theta R_{n+m}}\big) \leq \mathbb{E}\big(e^{\theta R_n + \theta R_m}\big) \leq \mathbb{E}\big(e^{\theta R_n}\big) \times \mathbb{E}\big(e^{\theta R_m}\big).$$

this implies that $\{\log \mathbb{E}\big(e^{\theta R_n}\big)\}_{n=1}^{\infty}$ is a subadditive sequence. Hence, it follows by the standard subadditivity lemma that

$$\lim_{n\to\infty} \Lambda_n(\theta) = \Lambda(\theta) := \inf_{n\geq 1} \Lambda_n(\theta). \tag{1.2.21}$$

Moreover, Hamana also showed the limit results for $\Lambda(\theta)$:

$$\lim_{\theta\downarrow 0} \frac{\Lambda(\theta)}{\theta} = \kappa, \qquad \lim_{\theta\to\infty} \frac{\Lambda(\theta)}{\theta} = 1.$$

Note that, by Theorem 1.2.3, we can deduce that $\Lambda(\theta) = 0$ for $\theta \leq 0$. Hence, we can conclude that, for $d \geq 3$, Λ is not differentiable at zero.

In both cases, since the constants in the exponent forms are both positive (compare with Theorem 1.2.3), this reflects the asymptotic behaviour in the upward direction. However, both results cannot be transformed in a standard large deviation set up, since the differentiabilities of $S(\lambda)$ and $\Lambda(\theta)$ are unknown, except at zero for $\Lambda(\cdot)$. Therefore, we cannot easily apply Gärtner- Ellis theorem.

However, Hamana and Kesten [HK01] show that the result can be written in large deviation form. This has been done by completely different technique.

Theorem 1.2.6. *The function*

$$\psi(\theta) = \lim_{n\to\infty} \frac{-1}{n} \log \mathbb{P}(R_n \geq \theta n)$$

exists for all θ, and satisfies

$$\begin{aligned} \psi(\theta) = 0, &\quad \textit{for } \theta \leq \kappa, \\ 0 < \psi(\theta) < \infty, &\quad \textit{for } \kappa < \theta \leq 1, \\ \psi(\theta) = \infty, &\quad \textit{for } \theta > 1. \end{aligned}$$

Moreover, in dimension two or more, $x \mapsto \psi(x)$ is continuous and convex on $[0,1]$ and strictly increasing on $[\kappa, 1]$.

The authors also prove the analogous result in a Wiener sausage case. For the Wiener sausage $W^a(t)$ with a constant drift μ (possibly zero),

$$\phi(x) := \lim_{t\to\infty} \frac{-1}{t} \log \mathbb{P}(|W^a(t)| \geq tx) \text{ exists} \tag{1.2.22}$$

for all $x \in \mathbb{R}$ and $\phi(\cdot) = 0$ for $x \leq \kappa_a$. The exact form for the rate functions $\psi(x)$ and $\phi(x)$ are still open questions.

Remark: Apart from $\theta \in (\kappa, 1]$, it seems to be very trivial. Indeed, we knew that R_n is bounded above by n, hence $\mathbb{P}(R_n \geq \theta n) = 0$ for $\theta > 1$. Moreover, by Theorem 1.2.1 we can deduce that

$$\lim_{n\to\infty} \mathbb{P}(R_n \geq \theta n) = 1 \quad \text{ for all } \theta < \kappa.$$

Therefore, the only interesting case is when $\kappa < \theta < 1$, the case which the random walk satisfies a large deviation principle with speed n and positive rate function $\psi(\theta)$.

Proof strategy. We repeat the words used by the authors of [HK01].

> The proof bases on an approximation on a subadditivity relation. We build a path of length $n + m$ with $R_{n+m} \geq y + z - E(n, m)$ for some error term $E(n, m)$ from the two paths $\mathfrak{P}_1$ and $\mathfrak{P}_2$. The error term comes from the fact that some points are counted in the range of both $\mathfrak{P}_1$ and $\mathfrak{P}_2$. Note that $\mathfrak{P}_1$($\mathfrak{P}_2$, respectively) has length $n(m)$ and range greater than or equal to $y(z)$. In order to make small overlap, we do not count the initial point of $\mathfrak{P}_2$ and the endpoint of $\mathfrak{P}_1$. The initial point of $\mathfrak{P}_2$ shall be placed at the distance at most of order $nm^{1/(d+1)}$ from the endpoint of $\mathfrak{P}_1$ in order to get the error term of order $(nm)^{1/(d+1)}$. The two paths are then connected at not too large cost in probability. This results in the inequality:
>
> $$\begin{aligned}\mathbb{P}(R_{n+m} &\geq y + z - (2d + 2)(nm)^{1/(d+1)}) \\ &\geq \frac{1}{2}\zeta^{d(nm)^{1/(d+1)}+d}\mathbb{P}(R_n \geq y)\mathbb{P}(R_m \geq z),\end{aligned}$$
>
> for some $\zeta > 0$.
> Note that when $d \geq 2, (nm)^{1/(d+1)}$ is small with respect to $\max\{n, m\}$. From this, we can use more or less standard subadditivity argument that the limit $\psi(x)$ exists at all continuity points $x \in (0, 1)$ of
>
> $$\underline{\psi(x)} := \liminf_{n\to\infty} \frac{-1}{n} \log \mathbb{P}(R_n \geq \theta n).$$
>
> It is then easy to obtain from the equation that the restriction of $\psi(x)$ to

the continuity points of ψ in $(0,1)$ is convex. This is enough to conclude that ψ is in fact continuous on $(0,1)$, just as in the usual proof of continuity of a convex function. Hence $\psi(x)$ exists for all $(0,1)$.

$\square$

Summary and deviation on the other scales

In brief, we summarise the known results so far on behaviours of the range. Jain and Pruitt [JP71] showed that $R_n - \mathbb{E}R_n$ satisfies central limit theorem on the scale $\sqrt{n}$ for $d \geq 4$, and on the scale $\sqrt{n \log n}$ for $d = 3$. For the scale $n^{d/(d+2)}$, Donsker and Varadhan [DV75, DV79] showed that both R_n and $|W^a(t)|$ in the *downward* direction satisfy the large deviation principle with speed $n^{d/(d+2)}$ and the same explicitly given rate function. Then, for the mean scale n, Hamana and Kesten [HK01] showed that the behaviours of both R_n and $|W^a(t)|$ in the *upward* direction satisfy the large deviation principle with speed n but with an unknown rate function. Also on this same scale, van den Berg, Bolthausen and den Hollander [BBH01] show the large deviation behaviour in the *downward* direction for $|W^a(t)|$ with speed $n^{(d-2)/d}$ and an explicitly given rate function.

Note that, the behaviour of the range on the scale $b_n = n^{d/(d+2)}$ in upward direction is still unknown. The results by Hamana in (1.2.20) and (1.2.21) do not help in this case. Even if we assume that $\Lambda(\theta)$ is differentiable (hence we can use Gärtner- Ellis theorem), the transformation only give the similar type as in Theorem 1.2.6. In order to get the large deviation of the scale $n^{d/(d+2)}$, we need a result of the type:

$$\lim_{n\to\infty} \frac{1}{n^{d/(d+2)}} \log \mathbb{E} \exp(\theta R_n), \qquad \theta > 0.$$

However, by (1.2.20) and (1.2.21), we can deduce that:

$$\lim_{n\to\infty} \frac{1}{n^{d/(d+2)}} \log \mathbb{E} \exp(\theta R_n) = \infty.$$

Moreover, Bass and Kumagai [BK02] (also, see Theorem 8.5.2 [Che10]) shows the moderate deviation for $R_n - \mathbb{E}R_n$ in $d = 3$

Theorem 1.2.7. *Let $d = 3$ and C is a constant. For any $\lambda > 0$ and positive sequence c_n satisfying $\lim_{n\to\infty} c_n = \infty$ and $c_n = o(\sqrt{\log n})$,*

$$\lim_{n\to\infty} \frac{1}{c_n} \log \mathbb{P}\Big\{ \pm (R_n - \mathbb{E}R_n) \geq \lambda \sqrt{n c_n \log n} \Big\} = -\frac{C\lambda^2}{\kappa^4}.$$

Note that the form in the theorem come from Theorem 8.5.2 [Che10]. Therefore, when we consider the problem in both downward and upward directions on the scale b_n, Theorem 1.2.7 suggests that the moderate deviation principle is valid when

$$\sqrt{n \log n} < b_n < (\log n)^{1/4} \sqrt{n \log n},$$

with speed c_n and rate function $C\lambda^2/\kappa^4$.

Now, an open problem would be to complete all the gap for the scale $b_n \in (\sqrt{n}, n)$. To do this, define

$$f_n \ll g_n \text{ if and only if } \lim_{n\to\infty} \frac{f_n}{g_n} = 0, \tag{1.2.23}$$

and similarly for $g_n \gg f_n$. Note that for a problem on the scale $b_n \gg n$ is not interesting for both downward and upward directions since the behaviour becomes trivial by the law of large number and the discrete property of R_n. For the rest, we give some conjectures:

1. $\sqrt{n} < b_n < n^{d/(d+2)}$ for $d \geq 4$: This can be suspected that R_n satisfies the *moderate deviation principle*.
2. $n^{d/(d+2)} < b_n < n$ for $d \geq 3$: It is reasonable to believe that $R_n - \mathbb{E}R_n$ satisfies the *large deviation principle* for this scale since both the upper and lower bound of b_n satisfies the large deviation principle.

We describe conjectures on these scales below.

Moderate deviation on R_n for $d \geq 4$

Note that the proof from Theorem 1.2.7 relies on the integrability property of random walks in dimension three. Hence, the proof can not be carried out in dimension four or more. We, however, suspect that the moderate deviation principle can also be carried out it the same way as in Theorem 1.2.7, and the conjecture for the moderate deviation principle for $d \geq 4$ is given by Chen (Conjecture 8.7.1, [Che10]).

Conjecture 1.2.8. *Let $d \geq 4$. For any $\lambda > 0$ and a_n satisfying $\lim_{n\to\infty} a_n = \infty$ and $a_n = o(n^{\frac{d-2}{d+2}})$,*

$$\lim_{n\to\infty} \frac{1}{a_n} \log \mathbb{P}(\pm(R_n - \mathbb{E}R_n) \geq \lambda\sqrt{na_n}) = -\frac{\lambda^2}{2\tilde{\mathfrak{D}}^2},$$

where $\tilde{\mathfrak{D}}^2$ is the variance of R_n described in Theorem 1.2.2.

Note that when $a_n = n^{\frac{d-2}{d+2}}$, this is exactly the same scale as in Donsker-Varadhan result in Theorem 1.2.3. Therefore, the sequence a_n must be less than this scale. This conjecture implies that the moderate deviation valid with speed a_n and the rate function $\lambda^2/2\tilde{\mathfrak{D}}^2$. A partial result has been obtained in Theorem 8.7.2 of [Che10]. They showed that

$$\limsup_{n\to\infty} \frac{1}{a_n} \log \mathbb{P}(R_n - \mathbb{E}R_n \geq \lambda\sqrt{na_n}) \leq -\frac{\lambda^2}{2\tilde{\mathfrak{D}}^2},$$

for $\lambda > 0$ and positive sequence a_n satisfying $a_n \to \infty$ and $a_n = o(n)$ as $n \to \infty$.

Large deviation on R_n on the scale between $n^{\frac{d}{d+2}}$ and n

We point out a conjecture made by Chen (Conjecture 8.7.3 [Che10]) to show the large deviation for the centred sequence.

Conjecture 1.2.9. *Let $d \geq 4$. For any $\lambda > 0$ and b_n satisfying $\lim_{n\to\infty} b_n/n^{\frac{d-2}{d+2}} = \infty$ and $b_n = o(n^{\frac{d-2}{d}})$,*

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}(R_n - \mathbb{E}R_n \leq -\lambda b_n^{\frac{d}{d-2}}) = \tilde{\mathcal{C}}(s)\lambda^{\frac{d-2}{d}}, \tag{1.2.24}$$

where $\tilde{\mathcal{C}}(s)$ is a constant.

Note that, we will not study this conjecture in this thesis. However, we explain here where the conditions of b_n come from:

- *Upper range* $b_n = o(n^{(d-2)/d})$: This come from the large deviation for the range on the mean scale. This is pretty obvious since by replacing b_n by $n^{(d-2)/d}$, the corresponding event will be $\{R_n \leq \mathbb{E}R_n - \lambda n\}$ which is the same type of the main problem of this thesis, and hence satisfies the large deviation principle with speed $n^{(d-2)/d}$.
- *Lower range* $b_n \gg n^{\frac{d-2}{d+2}}$: This comes from the Donsker-Varadhan result in Theorem 1.2.3, as well as Conjecture 1.2.8 on the moderate deviation principle.

Table 1.1 shows the summary of the behaviours of $R_n - \mathbb{E}R_n$ with various scales b_n in downward $\big(\mathbb{P}(R_n - \mathbb{E}R_n \leq b_n)\big)$ and upward $\big(\mathbb{P}(R_n - \mathbb{E}R_n \geq b_n)\big)$ directions for $d = 3$. The only exception case would be in Donsker-Varadhan case for the scale $b_n = n^{d/(d+2)}$, the result is for non-centred sequence, i.e. $\mathbb{P}(R_n \leq b_n)$.

Also, Table 1.2 shows the summary of the behaviours of $R_n - \mathbb{E}R_n$ with various scales for $d \geq 4$. Similarly, for the scale $b_n = n^{d/(d+2)}$, the result is also for non-centred sequence.

Table 1.1: Behaviours of $R_n - \mathbb{E}R_n$ on various scales for d= 3.

Scale	Downward direction	Upward direction
$b_n = (n \log n)^{1/2}$	Central limit theorem ([JP71])	
$(n \log n)^{1/2} < b_n < n^{1/2}(\log n)^{3/4}$	Moderate deviation ([BK02])	
$n^{1/2}(\log n)^{3/4} < b_n < n^{d/(d+2)}$	-	
$b_n = n^{d/(d+2)}$	Large deviation[1]: speed $n^{d/(d+2)}$ ([DV79])	-
$n^{d/(d+2)} < b_n < n$	Large deviation (conjecture) -	
$b_n = n$	Large deviation: speed $n^{(d-2)/d}$ (Theorem 1.2.10)	Large deviation: speed n ([HK01])

Table 1.2: Behaviours of $R_n - \mathbb{E}R_n$ on various scales for d $\geq$ 4.

Scale	Downward direction	Upward direction
$b_n = n^{1/2}$	Central limit theorem ([JP71])	
$n^{1/2} < b_n < n^{d/(d+2)}$	Moderate deviation (conjecture) -	
$b_n = n^{d/(d+2)}$	Large deviation[1]: speed $n^{d/(d+2)}$ ([DV79])	-
$n^{d/(d+2)} < b_n < n$	Large deviation (conjecture) -	
$b_n = n$	Large deviation: speed $n^{(d-2)/d}$ (Theorem 1.2.10)	Large deviation: speed n ([HK01])

[1] Large deviation on event $\{R_n \leq n^{d/(d+2)}\}$.

1.2.2 Main results for the ranges

In this section, we show our main result in full details, and give comments for the result. We remind that κ is the non-return probability of a random walk defined in (1.2.2).

Theorem 1.2.10. *Let $d \geq 3$. For every $b > 0$,*

$$\lim_{n\to\infty} \frac{1}{n^{\frac{d-2}{d}}} \log \mathbb{P}(R_n \leq bn) = -\frac{1}{d} I^{\kappa}(b), \tag{1.2.25}$$

where

$$I^{\kappa}(b) = \inf_{\phi\in\Phi^{\kappa}(b)} \Big[\frac{1}{2}\int_{\mathbb{R}^d} |\nabla\phi|^2(x)\,dx\Big] \tag{1.2.26}$$

with

$$\Phi^{\kappa}(b) = \Big\{\phi \in H^1(\mathbb{R}^d)\colon \int_{\mathbb{R}^d} \phi^2(x)dx = 1, \int_{\mathbb{R}^d} (1 - e^{-\kappa\phi^2(x)})dx \leq b\Big\}. \tag{1.2.27}$$

Remarks:

1. This is a large deviation result for the range of random walk when it deviates on its mean in the downward direction. This is an analogous result of that in Wiener sausage proved by [BBH01].

2. Apart from the factor $1/d$ in (1.2.25), the rate function coincides with the rate function of Wiener sausage case in (1.2.17)-(1.2.19) except that κ_a is replaced by κ. This is due to the local central limit theorem which we will describe in Remark 4.

3. The optimal strategy for the random walk is identical to that in Wiener sausage, this is to push the random walk towards the origin. On the event $\{R_n \leq b_n\}$, the walk behaves like the walk on a drift field $xn^{1/d} \mapsto (\nabla\phi/\phi)(x)$ for some smooth $\phi : \mathbb{R}^d \to [0, \infty)$. Conditioned on this drift, the walk spends time $\phi^2(x)$ in the box of size $Nn^{1/d}$ and its range only cover a fraction of $1 - e^{-\kappa\phi^2(x)}$ of the box. This strategy is more complicated than that in Donsker-Varadhan result [DV79] at which the optimal strategy is to stay inside a ball and fill all the space until time n. The cost of this strategy made an effect on the speed and the rate function of the large deviation event. It seems that the second constraint in (1.2.27) is the main condition for the rate function. This will be explained in the remark of Proposition 1.2.11.

4. The structure of the proof is also similar to that in Wiener sausage case in [BBH01].

However, there are some additional technical difficulties centred around the following three topics:

- *Local central limit theorem*: In the course of the argument, we have to work with discrete probability transition density which by a local central limit theorem converge to the transition densities of Brownian motion. This approximation leads to additional error terms and $n-$dependencies, which need to be controlled.
- *Potential theory*: The potential theoretic case requires a significant change from the set up of [BBH01]. The path reversal argument becomes more transparent in the discrete case and the reasons for the occurrence of the non-return probability κ are worked out clearly.
- *Large deviation principle*: The classical Donsker-Varadhan argument used for the LDP of the empirical pair measures needs to be strengthened to incorporate the passage from the discrete problem to a continuous rate functional. A parity issue also needs to be taken into account.

1.2.3 The outline

In this section, we give the outline for the proof of Theorem 1.2.10. The full details of the proof will be shown in Chapter 2.

First of all, we begin the section by doing standard compactification. For $N \in \mathbb{N}$ even, let $\Lambda_N = \big[-\frac{N}{2}, \frac{N}{2}\big)^d$ be the torus of size $N > 0$ with periodic boundary conditions. Throughout the proof of Proposition 1.2.11, which we will introduce later in this section, the random walk will live on $\Lambda_{Nn^{\frac{1}{d}}} \cap \mathbb{Z}^d$ with N fixed. We denote by $(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ the random walk on $\Lambda_{Nn^{\frac{1}{d}}} \cap \mathbb{Z}^d$ and by $\mathcal{R}_n$ the number of lattice sites visited by the random walk *on the torus* up to time $n \in \mathbb{N}$.

To prove Theorem 1.2.10, we will show that the upper bound and the lower bound of the left hand side of (1.2.25) are the required rate function, i.e. we show that:

$$\limsup_{n\to\infty} \frac{1}{n^{\frac{d-2}{d}}} \log \mathbb{P}(R_n \le bn) \le -\frac{1}{d} I^\kappa(b), \tag{1.2.28}$$

$$\liminf_{n\to\infty} \frac{1}{n^{\frac{d-2}{d}}} \log \mathbb{P}(R_n \le bn) \ge -\frac{1}{d} I^\kappa(b). \tag{1.2.29}$$

In order to deduce (1.2.28) and (1.2.29), we will given an introduction of the transition probability of a random walk in Section 2.1. This includes a property of the transition probability of the random walk on a torus. In Section 2.2, we deduce the large deviation principle of random walk on a torus. Finally we complete the proof of Theorem 1.2.10 by showing (1.2.28) and (1.2.29) in Section 2.3.

We now explain the main steps of Section 2.2 and Section 2.3.

Large deviation behaviour of random walk on torus

We show that the range of random walk wrapped around $\Lambda_{Nn^{1/d}}$ satisfies a large deviation principle.

Proposition 1.2.11. *Let $d \geq 3$, then $\frac{1}{n}\mathcal{R}_n$ satisfies a large deviation principle on $\mathbb{R}^+$ with speed $n^{\frac{d-2}{d}}$ and rate function $\frac{1}{d}J_N^\kappa$, where*

$$J_N^\kappa(b) = \inf_{\phi \in \partial\Phi_N^\kappa(b)} [\frac{1}{2}\int_{\Lambda_N} |\nabla\phi|^2(x)dx] \tag{1.2.30}$$

with

$$\partial\Phi_N^\kappa(b) = \{\phi \in H^1(\Lambda_N) : \int_{\Lambda_N} \phi^2(x)dx = 1, \int_{\Lambda_N} \big(1 - e^{-\kappa\phi^2(x)}\big)dx = b\}. \tag{1.2.31}$$

Remarks:

1. Proposition 1.2.11 implies the following:

   Corollary 1.2.12. *Let $d \geq 3$. For every $b > 0$ and $N > 0$,*

   $$\lim_{n\to\infty} \frac{1}{n^{\frac{d-2}{d}}} \log \mathbb{P}\big(\tfrac{1}{n}\mathcal{R}_n \leq b\big) = -\tfrac{1}{d}I_N^\kappa(b), \tag{1.2.32}$$

   where $I_N^\kappa(b)$ is given by the same formula as in (1.2.26) and (1.2.27) except that $\mathbb{R}^d$ is replaced by Λ_N.

2. Observe that the main condition for the rate function is the third term of (1.2.27). Clearly,

   $$b = \int_{\mathbb{R}^d} (1 - e^{-\kappa\phi^2(x)})dx \leq \int_{\mathbb{R}^d} \kappa\phi^2(x)dx = \kappa.$$

   This implies that the rate function is infinite for $b > \kappa$. This agree with the result in Theorem 1.2.1

In order to prove Proposition 1.2.11, we divide the proof into four main steps.

- Section 2.2.1: For $\epsilon > 0$, define the skeleton walk

$$\mathbb{S}_{n,\epsilon} = \{\mathcal{S}_{i\epsilon n^{\frac{2}{d}}}\}_{1 \leq i \leq \frac{1}{\epsilon}n^{\frac{d-2}{d}}}. \tag{1.2.33}$$

Using this skeleton walk, we approximate $\frac{1}{n}\mathcal{R}_n$ by $\mathbb{E}_{n,\epsilon}\frac{1}{n}\mathcal{R}_n$, where $\mathbb{E}_{n,\epsilon}$ denotes the conditional expectation given $\mathbb{S}_{n,\epsilon}$. By application of a concentration inequality of Talagrand, the difference between $\frac{1}{n}\mathcal{R}_n$ and $\mathbb{E}_{n,\epsilon}\frac{1}{n}\mathcal{R}_n$ is negligible in the limit as $n \to \infty$ followed by $\epsilon \downarrow 0$.

- Section 2.2.2: We represent $\mathbb{E}_{n,\epsilon}\frac{1}{n}\mathcal{R}_n$ as a continuous function of the pair empirical measure:

$$L_{n,\epsilon} = \epsilon n^{-\frac{d-2}{d}} \sum_{i=1}^{\frac{1}{\epsilon}n^{(d-2)/d}} \delta_{(n^{-1/d}\mathcal{S}_{(i-1)\epsilon n^{2/d}}, n^{-1/d}\mathcal{S}_{i\epsilon n^{2/d}})}. \tag{1.2.34}$$

  This will be done in (2.2.32) and it proved to be a key step for the proof. This is because, by a variant of the classical Donsker-Varadhan theory, $(L_{n,\epsilon})_{n>0}$ satisfies a LDP. We will get LDP for $\mathbb{E}_{n,\epsilon}\frac{1}{n}\mathcal{R}_n$ via the contraction principle.

- Section 2.2.3: Finally, we perform the limit $\epsilon \downarrow 0$. By the result from Section 2.1.1, we already know that $\frac{1}{n}\mathcal{R}_n$ is well approximated by $\mathbb{E}_{n,\epsilon}\frac{1}{n}\mathcal{R}_n$. It therefore suffices to have an appropriate approximation for the variational formula in the LDP for $\mathbb{E}_{n,\epsilon}\frac{1}{n}\mathcal{R}_n$.

Then, we collect all the results from Section 2.2.1 to Section 2.2.3 and complete the proof of Proposition 1.2.11 in Section 2.2.4.

Sketch of the proof of Theorem 1.2.10

We give the outlines of the section:

- To prove (1.2.28), our main steps are the following:
  1. We project the walk on $\mathbb{Z}^d$ to a torus $\Lambda_{Nn^{1/d}}$. This is done because we need to use the fact that $\mathcal{R}_n \leq R_n$,which will give the upper bound in probability for our event.
  2. We then instead prove the large deviation principle for the random walk on the torus.
  3. Finally, we increase the size of torus and get the required rate function in (1.2.25) by Proposition 1.2.13.
- To show (1.2.29), we will do the following: We will again let the random walk lives on $\Lambda_{Nn^{1/d}}$ and we let $C_{Nn^{1/d}}(n)$ to be the event that the random walk will

not hit the boundary of $\Lambda_{Nn^{1/d}}$ until time n. Then, we find the lower bound in probability of the event $\{R_n \le bn\}$ by considering the inequality

$$\mathbb{P}(R_n \le bn) \ge \mathbb{P}\big(\mathcal{R}_n \le bn, C_{Nn^{1/d}}(n)\big). \qquad (1.2.35)$$

Then, we deduce that the right hand side of (1.2.35) give an appropriate limit which converges to the required limit in (1.2.25) when we increase the size of the torus.

Note that, in order to complete the proofs of (1.2.28) and (1.2.29), we need a result to show that when the size of torus increases, the rate function for the random walk on torus converges to the required rate function in (1.2.25). We denote $I_N^{\kappa}(b)$ is a rate function on Λ_N given by the same formula as in (1.2.26) and (1.2.27) except that $\mathbb{R}^d$ is replaced by Λ_N.

Proposition 1.2.13. $\lim_{N\to\infty} I_N^{\kappa}(b) = I^{\kappa}(b)$ *for all* $b > 0$ *where*

- $I_N^{\kappa}(b)$ *is given by the same formula as in (1.2.26) and (1.2.27) except that* $\mathbb{R}^d$ *is replaced by* Λ_N*.*
- I^{κ} *is the rate function defined in Theorem 1.2.10.*

1.3 The intersections

In this section, we focus on problems and developments on the intersections of ranges of independent random walks. Firstly, we give general overviews and known results in Section 1.3.1. Then, we present our main result and give extra comments in Section 1.3.2. Finally, the outline of the proof of our result is explained in Section 1.3.3. We remind that J_n^k is the intersection of independent ranges of k random walks defined in (1.1.2) and J_n is defined in (1.1.3). Also, $|V_k^a(t)|$ is the intersection volume of k independent Wiener sausage with radius a defined in (1.1.7) , with $V^a(t) := V_2^a(t)$. Also, we define *infinite-time intersection of the ranges of random walks* and, respectively, *infinite-time intersection volume of Wiener sausages* as:

$$J_\infty^k := \lim_{n\to\infty} J_n^k, \qquad |V_k^a| := \lim_{t\to\infty} |V_k^a(t)|, \tag{1.3.1}$$

with our usual notations $J_\infty := J_\infty^2$ and $|V^a| := |V_2^a|$.

1.3.1 Overview

We again start the section with an obvious fact that J_n is bounded, this is due to the discreteness property of random walk.

$$0 \le J_n \le n. \tag{1.3.2}$$

Remarks:

1. To get $J_n = n$, the only strategy is for one random walk to perform a self-avoiding walk and the other walk follows the first random walk at every step. Also, to get $J_n = 0$ is rather obvious, the random walks are not make any intersection at all during time n.

2. It is also easy to see that (1.3.2) can also be extended to J_n^k.

3. We note that the J_n is also bounded by each individual range, i.e.

$$J_n \le R_n^1, \qquad J_n \le R_n^2.$$

   This is pretty obvious, and it can be derived from (1.1.5).

4. There are also more complicated bounds for J_n, for the moments of J_n, for example, from Theorem 6.2.1 [Che10], when $d \ge 3$, there exists a constant $C > 0$

such that

$$(\mathbb{E}J_n)^m \le (m!)^{3/2} C^m n^{m/2}, \qquad m, n \in \mathbb{N}.$$

Before we show the result on expected value of J_n, we would like show the result described in Dvoretzky-Erdös [DE51] and Dvoretzky-Erdös-Kakutani [DEK50, DEK54]:

$$J_\infty^k = \infty \text{ a.s.} \quad \text{if and only if} \quad k(d-2) < d. \tag{1.3.3}$$

This result shows that k independent random walks will make finitely many intersections if $k < d/(d-2)$. Since we concentrate on the case $k = 2$, we can classify the dimensions into

- *Subcritical dimensions*: $d < 4$.
- *Critical dimension*: $d = 4$.
- *Supercritical dimensions*: $d > 4$.

As we may expect, J_n behaves differently in each case. Moreover, in the subcritical case, the behaviour of the intersections in one and two dimensions is different from three dimensions because of the recurrence property.

We now show the expected value of J_n in high dimensions. The result has been shown by Erdös and Taylor [ET60], and also by Le Gall [LeG86a, LeG86b]:

Theorem 1.3.1. *As $n \to \infty$,*

$$\mathbb{E}(J_n) = \begin{cases} c_3 n^{\frac{1}{2}}(1+o(1)), & d=3 \\ c_4 \log n(1+o(1)), & d=4 \\ c_d(1+o(1)), & d \ge 5 \end{cases}$$

where c_i is a finite positive constant for all i in $\mathbb{Z}^+$.

Proof(sketch). We only show this for $d = 3$, since the other dimensions follow the same method. Also, we only show the lower bound. Note that:

$$\begin{aligned} \mathbb{E}J_n &\ge \sum_{x \in \mathbb{Z}^d : |x| \le \sqrt{n}} \big(\mathbb{P}(S_i^1 = S_j^2 = x \text{ for some } 1 \le i, j \le n)\big), \\ &= \sum_{x \in \mathbb{Z}^d : |x| \le \sqrt{n}} \big(\mathbb{P}(S_i = x \text{ for some } 1 \le i \le n)\big)^2 \end{aligned} \tag{1.3.4}$$

gives the desired lower bound. Note that, for some constant c by, e.g.,[ET60]

$$\mathbb{P}(S_i = x \text{ for some } 1 \le i \le n) > \frac{c}{|x|^{d-2}}, \tag{1.3.5}$$

for $n > \frac{1}{5}|x|^2$. By using (1.3.4) and (1.3.5), we get

$$\mathbb{E}J_n \ge \sum_{\{r:1\le 2^r<\sqrt{n}\}} \sum_{\{x:2^{r-1}\le|x|\le 2^r\}} \left(\frac{c}{|x|^{d-2}}\right)^2 \ge \sum_{\{r:1\le 2^r<\sqrt{n}\}} 2^{dr} \times \left(\frac{c}{2^{r(d-2)}}\right)^2.$$

Let $d = 3$, and we can derive that $\mathbb{E}J_n \ge C\sqrt{n}$, hence, we get the desired lower bound. We will not do the proof for upper bound here, but the idea is to write $\mathbb{E}J_n$ as :

$$\begin{aligned}\mathbb{E}J_n = &\sum_{x\in\mathbb{Z}^d:|x|\le\sqrt{n}} \mathbb{P}(S_i^1 = S_j^2 = x \text{ for some } 1 \le i, j \le n)\\ &+ \sum_{x\in\mathbb{Z}^d:|x|>\sqrt{n}} \mathbb{P}(S_i^1 = S_j^2 = x \text{ for some } 1 \le i, j \le n).\end{aligned} \tag{1.3.6}$$

Then, the first term on the right hand side of (1.3.6) can be approximated as in the similar way as in the lower bound case, and it will give the upper bound of order $\sqrt{n}$. For the second term on the right hand side of (1.3.6), we need to show that this grows slower than $\sqrt{n}$. One way to show this is from the local central limit theorem (Lemma 2.1.1(a) in Chapter 2). □

Remarks:

1. We can see that for dimension five or more, two random walks make only finitely many intersections. This agrees with the result described in (1.3.3). The expectations of J_n behave differently in subcritical, critical and supercritical dimensions. Also, note that by comparing with Theorem 1.2.1 we can see that the expectation of J_n is smaller than the expectation of R_n.

2. We can find the weak law of J_n [LeG86a, LeG86b]:

$$\frac{J_n}{n} \xrightarrow{d} (\det\Gamma)^{-1/2}\kappa^2\alpha([0,1]^2), \qquad d = 3 \tag{1.3.7}$$

$$\frac{J_n}{\log n} \xrightarrow{d} \frac{\kappa^2}{4\pi^2}(\det\Gamma)^{-1/2}\mathcal{U}^2, \qquad d = 4, \tag{1.3.8}$$

   where Γ is the covariance matrix of a single random walk, $\mathcal{U}$ is a standard Gaussian random variable, and $\alpha([0,1]^2)$ is the Brownian intersection local time symbolically defined by:

$$\alpha([0,1]^2) = \int_{\mathbb{R}^d} \left[\prod_{j=1}^{2} \int_0^1 \delta_x\big(W_j(s)\big) ds \right] dx,$$

where $W_1(t)$ and $W_2(t)$ are independent 3-dimensional Brownian motions. The explanation of the Brownian intersection local time can be seen in [GHR84] for example.

3. We again show an analogous result for the intersection volume of independent Wiener sausages. Le Gall [LeG86b] obtained the result for the expectation of the intersection volume. Then, van den Berg [vdB05] gave the exact forms of the expectations. As $t \to \infty$ and $\hat{c}$ is a constant depends on dimension and the radius of the Wiener sausages,

$$\mathbb{E}|V^a(t)| = \begin{cases} \hat{c}_3 t^{1/2}(1+o(1)), & d=3, \\ \hat{c}_4 \log t(1+o(1)), & d=4, \end{cases} \tag{1.3.9}$$

and for $d \geq 5$,

$$\lim_{t\to\infty} \mathbb{E}|V^a(t)| = \hat{c}_d.$$

The exact values for the constants are shown in [vdB05, vdB11].

Now, we would like to introduce another intersection quantity. Define *the mutual intersection local time of k independent random walks* by:

$$I_n^k = \sum_{j_1,\ldots,j_k=1}^{n} \mathbf{1}\{S^1_{j_1} = \ldots = S^k_{j_k}\}, \tag{1.3.10}$$

with our usual set up $I_n := I_n^2$. Even though we are not concentrating on problems on I_n for this thesis, it suggests that there is a relation between I_n and J_n. We list some facts about I_n:

1. Observe that $0 \leq I_n^k \leq n^k$. Also $I_n^k \geq J_n^k$ which come from the fact that random walks may make multiple visits at intersection points.

2. It is obvious that I_n and J_n are related in a positive way, i.e. the more J_n, the more chance of intersection local time. Indeed, Le Gall and Rosen [LGR91] show that for $d=3$,

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\Big(I_n - \kappa^2 J_n\Big)^2 = 0, \tag{1.3.11}$$

where we remind that κ is the non-return probability defined in (1.2.2). However, it seems that the relation in (1.3.11) does not hold in the critical and supercritical dimensions.

3. With the help of (1.3.11) we can see that I_n and J_n converges to a limit in the same order. $\mathbb{E}I_n$ is of the same order as $\mathbb{E}J_n$. Also, I_n satisfies the same weak law as J_n with constant difference, i.e. for $d = 3$,

$$\frac{I_n}{n} \xrightarrow{d} (\det \Gamma)^{-1/2} \alpha([0,1]^2).$$

Note that the similar result also hold in the critical dimension, $d = 4$.

4. By the result in (1.3.3), we also have

$$I_\infty^k = \infty \text{ a.s.} \quad \text{if and only if} \quad k(d-2) < d. \tag{1.3.12}$$

Behaviours of J_n

We will show the results in subcritical dimension, critical dimension and supercritical dimensions respectively, since J_n behaves differently in each case.

1. For the subcritical dimension, we have a large deviation result for J_n by Chen [Che05].

Theorem 1.3.2. *For $d = 3$, a constant C_1 and any $\theta > 0$,*

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}(J_n \geq \theta\sqrt{nb_n^3}) = C_1\theta^{2/3}, \tag{1.3.13}$$

where b_n is any positive sequence satisfying $b_n \to \infty$ and $b_n = o(n^{1/3})$ as $n \to \infty$.

Remarks:

1. We have the restriction $b_n = o(n^{1/3})$ since $J_n \leq n$, which give zero probability in (1.3.13).

2. We also have an analogous result for I_n:

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}(I_n \geq \theta\sqrt{nb_n^3}) = C_1\theta^{2/3}, \tag{1.3.14}$$

where b_n is any positive sequence satisfying $b_n \to \infty$ and $b_n = o(n)$ as $n \to \infty$. The last restriction is different from J_n case because $I_n \leq n^2$.

2. For $d = 4$, we would expect the same behaviour as in the subcritical case. It has been studied by Marcus and Rosen [MR97] that

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}(J_n \geq \theta b_n \log n) = -\theta C_2(d,p), \tag{1.3.15}$$

for small scale $b_n(= o(\log\log\log n)$ in that paper). They also show that if b_n grows faster than $\log n$ will make (1.3.15) fail. Next, we show the conjecture made by Chen (Conjecture 7.4.2 [Che10]) to show the large deviation behaviour for J_n in the critical dimension:

Conjecture 1.3.3. *For $d = 4$, a constant C_3 and any $\theta > 0$,*

$$\lim_{n\to\infty} \frac{1}{b_n} \log \mathbb{P}(J_n \geq \theta b_n^2) = -C_3\theta^{1/2}, \tag{1.3.16}$$

where b_n is any positive sequence satisfying $\lim_{n\to\infty} \frac{b_n}{\log n} = \infty$ and $b_n = o(n^{1/2})$ as $n \to \infty$.

The conjecture shows that we might expect a similar behaviour as in (1.3.13) and (1.3.15) even if $b_n \gg \log n$. Note that since $J_n \leq n$, this restrict then b_n can not grow faster than $n^{1/2}$.

3. In supercritical dimensions, behaviour of I_n and J_n are completely different. Similar behaviour as in (1.3.11) does not appear. We confirm this remark by the work from Khanin, Mazel, Schlosman and Sinai [KMSS94] at which they study the tail distributions of the infinite-time intersections I_∞ and J_∞: For $d \geq 5$, there are $c_1, c_2 > 0$ such that

$$\exp(-c_1 t^{1/2}) \leq \mathbb{P}(I_\infty \geq t) \leq \exp(-c_2 t^{1/2}), \tag{1.3.17}$$

and that given $\delta > 0$,

$$\exp(-t^{\frac{d-2}{d}+\delta}) \leq \mathbb{P}(J_\infty \geq t) \leq \exp(-t^{\frac{d-2}{d}-\delta}) \quad \forall t \geq t_0. \tag{1.3.18}$$

holds for some t_0. The difference in behaviour of I_n and J_n comes from their difference in optimal strategies to get large values. In order to get I_∞ large, we let random walks stay inside a big but fixed ball and repeat the intersection at the same site. While for J_∞, this strategy does not work since they have to intersect at many different sites.

Indeed, the result for the infinite-time intersection local time has been shown by Chen and Mörters [CM09], and it follows (1.3.17).

Theorem 1.3.4. *For $d \geq 5$,*

$$\lim_{n\to\infty} \frac{1}{n^{1/2}} \log \mathbb{P}(I_\infty > n) = -2\mathcal{C}, \tag{1.3.19}$$

where $\mathcal{C}$ is positive and finite.

However, for the infinite-time intersection of the ranges, this is still an open problem. It can be guessed that we suspect a similar type of behaviour as in Theorem 1.3.4 for J_n, i.e. for some $\mathcal{I}$ which is positive and finite:

$$\lim_{n\to\infty} \frac{1}{n^{(d-2)/d}} \log \mathbb{P}(J_\infty > n) = -\mathcal{I}.$$

Finally, van den Berg, Bolthausen and den Hollander [BBH04] conjecture the result for J_∞ based on their work in the large deviation result for *the finite-time intersection volume of Wiener sausages* from Theorem 1.3.5 below. We remind that κ_a is the Newtonian capacity of $B_a(0)$.

Theorem 1.3.5. *Let $d \geq 3$. Then, for every $c > 0$,*

$$\lim_{t\to\infty} \frac{1}{t^{\frac{d-2}{d}}} \log \mathbb{P}(|V^a(ct)| \geq t) = -\hat{I}_d^{\kappa_a}(c), \tag{1.3.20}$$

where

$$\hat{I}_d^{\kappa_a}(c) = c \inf_{\phi\in\Phi_d^{\kappa_a}(c)} [\int_{\mathbb{R}^d} |\nabla\phi|^2(x)dx], \tag{1.3.21}$$

with

$$\Phi_d^{\kappa_a}(c) = \{\phi \in H^1(\mathbb{R}^d) : \int_{\mathbb{R}^d} \phi^2(x)dx = 1, \int_{\mathbb{R}^d} (1 - e^{-\kappa_a c\phi^2(x)})^2 dx \geq 1\}. \tag{1.3.22}$$

Remarks:

1. Our main result is obtaining an analogous result for the similar problem for the intersections of random walks. This will be explored in Section 1.3.2.
2. The result describes the tail distribution of infinite-time intersection of the ranges in continuous space-time setting but only after restricting the time horizon to a multiple of t. The authors pick a time horizon of length ct and are letting $t \to \infty$ for fixed $c > 0$. The size of the large deviation t come from the expected volume of a single Wiener sausage as $t \to \infty$. So, the two Wiener sausages are doing large deviation on the scale of their mean.
3. We repeat the words described for the optimal strategy of the proof from the

authors of [BBH04]:

> The idea behind the theorem is that the optimal strategy for the two Brownian motion to realise the large deviation event $\{|V^a(ct) \geq t|\}$ is to behave like Brownian motions in a drift field $xt^{1/d} \mapsto (\nabla\phi/\phi)(x)$ for some smooth $\phi : \mathbb{R}^d \to [0, \infty)$ during the given time window [0, ct]. Conditioned on adopting this drift, each Brownian motion spends time $c\phi^2(x)$ per unit volume in the neighbourhood of $xt^{1/d}$, thus using up total time $t\int_{\mathbb{R}^d} c\phi^2(x)dx$. This time must equal ct. Also, each corresponding Wiener sausage covers a fraction $1 - e^{-\kappa_a c\phi^2(x)}$ of the space in the neighbourhood of $xt^{1/d}$, thus making a total intersection volume $t\int_{\mathbb{R}^d}(1 - e^{-\kappa_a c\phi^2(x)})^2 dx$. This volume must exceed t. The cost for adopting the drift during time ct is $t^{(d-2)/d}\int_{\mathbb{R}^d} c|\nabla\phi|^2(x)dx$. The best choice of the drift field is therefore given by minimisers of the variational problem in (1.3.21) and (1.3.22).
>
> Note that the optimal strategy for the two Wiener sausage is to form a Swiss cheese: they cover only part of the space, leaving random holes whose size are of order 1 and whose density varies on space scale $t^{1/d}$. The local structure of this Swiss cheese depends on a. Also note that the two Wiener sausages follow the optimal strategy *independently*. Under the joint optimal strategy the two Brownian motions are independent on space scale smaller that $t^{1/d}$.

4. The result can be extended for the similar problem on three or more Wiener sausages, see Section 1.6 of [BBH04]. For the intersection volume $|V_k^a(t)|$, the results will be similar as in Theorem 1.3.5 except that c is replaced by $kc/2$ in (1.3.21) and $\int_{\mathbb{R}^d}(1 - e^{-\kappa_a c\phi^2(x)})^2 dx$ is replaced by $\int_{\mathbb{R}^d}(1 - e^{-\kappa_a c\phi^2(x)})^k dx$ in (1.3.22).

The authors describe a conjecture for the large deviation behaviour for the infinite-time intersection volume and we make a summary of the results here. Firstly, they get rid of the dependence of a and c by the following: Let $d \geq 2$ and $a > 0$. For every $c > 0$,

$$\hat{I}_d^{\kappa_a}(c) = \frac{1}{\kappa_a}\Theta_d(\kappa_a c), \tag{1.3.23}$$

where $\Theta_d : (0, \infty) \to [0, \infty]$ is given by

$$\Theta_d(u) = \inf\left\{||\nabla\psi||_2^2 : \psi \in H^1(\mathbb{R}^d), ||\psi||_2^2 = u, \textstyle\int(1 - e^{\psi^2})^2 \geq 1\right\}. \tag{1.3.24}$$

Also, we define $u_\diamond = \min_{\xi>0}\xi(1-e^{-\xi})^{-2}$. Then $\Theta_d(u)$ has the nice following properties:

- $\Theta_d = \infty$ on $(0, u_\diamond]$ and $0 < \Theta_d < \infty$ on $(u_\diamond, \infty)$.
- Θ_d is nonincreasing and continuous on $(u_\diamond, \infty)$. Also, $\Theta_d \asymp (u - u_\diamond)^{-1}$ as $u \downarrow u_\diamond$.
- For $2 \leq d \leq 4$, the mapping $u \mapsto u^{(4-d)/d}\Theta_d(u)$ is strictly decreasing on $(u_\diamond, \infty)$ and

$$\lim_{u\to\infty} u^{(4-d)/d}\Theta_d(u) = \mu_d,$$

  where $0 < \mu_d < \infty$.
- For $d \geq 5$, define

$$\eta_d = \inf\left\{||\nabla\psi||_2^2; \psi \in D^1(\mathbb{R}^d), \textstyle\int(1 - e^{\psi^2})^2 \geq 1\right\},$$

  then the exists a minimiser ψ_d of the variational problem. Moreover $||\psi_d||_2^2 < \infty$. Next, define $u_d = ||\psi_d||_2^2$. Then the mapping $u \mapsto \theta_d(u)$ is strictly decreasing on $(u_\diamond, u_d)$ and $\Theta_d(u) = \eta_d$ on $[u_d, \infty)$.
- Let $2 \leq d \leq 4$ and $u \in (u_\diamond \infty)$ or $d \geq 5$ and $u \in (u_\diamond, u_d]$. Then, the variational problem in (1.3.24) has a minimiser. There is no minimiser when $d \geq 5$ and $u \in (u_d, \infty)$.

We show the picture produced by van den Berg et al. in pp. 746 [BBH04] in Figure 1-1 to see the overall picture of Θ_d:

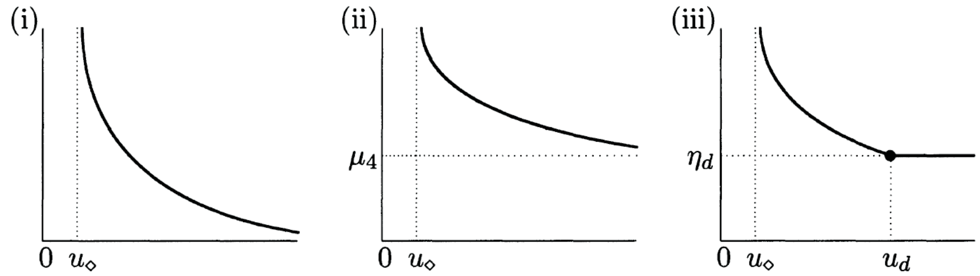

Figure 1-1: Qualitative picture of Θ_d made by van den Berg et al. for: (i) $d = 2, 3$; (ii) $d = 4$; (iii) $d \geq 5$.

Note that, although $\hat{I}_d^{\kappa_a}(c) = \hat{I}_d^{\kappa_a}(c_0)$ for all $c > c_0$ in $d \geq 5$ (see Figure 1-1 (iii)), it is not obvious to get the result for J_∞ from Theorem 1.3.5. Since it is not clear that the limit $t \to \infty$ and $c \to \infty$ can be interchanged. The intersection volume might prefer to exceed the value t on a timescale of order larger than t. Nevertheless, they suggest the conjecture for the infinite-time intersection:

Conjecture 1.3.6. *For $d \geq 5$,*

$$\lim_{t\to\infty} \frac{1}{t^{(d-2)/d}} \log \mathbb{P}(|V^a| \geq t) = -\hat{I}_d^{\kappa_a},$$

where

$$\hat{I}_d^{\kappa_a} = \inf_{c>0} \hat{I}_d^{\kappa_a}(c^*) = \frac{\eta_d}{\kappa_a},$$

with $c^* = u_d/\kappa_a$.

Also, it can be suggested from Figure 1-1 that the limits $t \to \infty$ and $c \to \infty$ can be interchanged for $d \geq 5$, but not for the dimensions four or less. We can also see that for $2 \leq d \leq 4$, the optimal strategy for the time horizon is for $c = \infty$. It is conjectured that the optimal strategy for $d \geq 5$ is similar to that in Theorem 1.3.5 for the finite-time intersection, we apply the drift for both Brownian motions up to time c^*t. After this time, we let the Brownian motions behave normally, which make them travel to infinity in different directions. This coincides with Figure 1-1(iii) where the function is a constant after the critical time.

1.3.2 Main results for the intersections

In this section, we show our main result in full details, and give comments for the result. We remind that κ is the non-return probability of a random walk defined in (1.2.2).

Theorem 1.3.7. *Let $d \geq 3$. Then, for every $a > 0$,*

$$\lim_{n\to\infty} \frac{1}{n^{(d-2)/d}} \log \mathbb{P}(J_n \geq an) = -\frac{1}{d} L^\kappa(a), \tag{1.3.25}$$

where

$$L^\kappa(a) = \inf_{\phi \in \hat{\Phi}^\kappa(a)} \Big[\int_{\mathbb{R}^d} |\nabla \phi|^2(x) dx \Big], \tag{1.3.26}$$

with

$$\hat{\Phi}^\kappa(a) = \{\phi \in H^1(\mathbb{R}^d) : \int_{\mathbb{R}^d} \phi^2(x) dx = 1, \int_{\mathbb{R}^d} \Big(1 - e^{-\kappa\phi^2(x)}\Big)^2 dx \geq a\}. \tag{1.3.27}$$

Remarks:

1. This is a large deviation result for the intersections of ranges of the two random walks in the upward direction. This is an analogous result of that in Wiener sausages proved by [BBH04] with essentially the same method. Note that, we can write (1.3.25) as:

$$\lim_{n\to\infty} \frac{1}{n^{(d-2)/d}} \log \mathbb{P}(J_n \geq an) = -\frac{a^{(d-2)/d}}{d} \hat{I}_d^\kappa\big(\tfrac{1}{a}\big), \tag{1.3.28}$$

   where $\hat{I}_d^\kappa$ is the same rate function as in Wiener sausages case defined in (1.3.21) except that κ_a is replaced by the non-return probability κ. Therefore, the conjecture for the large deviation behaviour for the infinite-time intersection also valid in the random walks case. We will show (1.3.28) later in this section.

2. Similar to a single random walk case, the extra factor $1/d$ enters the rate function in (1.3.25), and also the capacity of the ball κ_a is replaced by the non-return probability κ due to the local central limit theorem as in the problem on the range of a single random walk.

3. The optimal strategy is similar to that in Theorem 1.3.5. On the event $\{J_n \geq an\}$, the walks behave like the walks on a drift field $xn^{1/d} \mapsto (\nabla\phi/\phi)(x)$ for some smooth $\phi : \mathbb{R}^d \to [0, \infty)$. Conditioned on this drift, each walk spend time $\phi^2(x)$ in the box of size $Nn^{1/d}$ and its range only cover a fraction of $1 - e^{-\kappa\phi^2(x)}$ of the box. Moreover, each random walk follow this optimal strategy *independently.*

4. We can also obtain a result for the intersection of ranges of three or more random walks. This is pretty clear since each random walk follows the optimal strategy independently.

   Corollary 1.3.8. *Let $d \geq 3$ and $k \geq 3$. Then, for every $a > 0$,*

$$\lim_{n\to\infty} \frac{1}{n^{(d-2)/d}} \log \mathbb{P}(J_n^k \geq an) = -\frac{1}{d}\tilde{L}^\kappa(a), \tag{1.3.29}$$

   where

$$\tilde{L}^\kappa(a) = \frac{k}{2} \inf_{\phi\in\Psi^\kappa(a)} \Big[\int_{\mathbb{R}^d} |\nabla\phi|^2(x)dx \Big], \tag{1.3.30}$$

   with

$$\Psi^\kappa(a) = \{\phi \in H^1(\mathbb{R}^d) : \int_{\mathbb{R}^d} \phi^2(x)dx = 1, \int_{\mathbb{R}^d} \Big(1 - e^{-\kappa\phi^2(x)}\Big)^k dx \geq a\}. \tag{1.3.31}$$

   The proof will require minor modifications from the proof of Theorem 1.3.7.

Comparison of the rate functions

We show that the rate function in (1.3.26) can be written in the form of the rate function $\hat{I}_d^\kappa$, where

$$\hat{I}_d^\kappa(c) = c \inf_{\varphi\in\tilde{\Phi}_d^\kappa(c)} [\int_{\mathbb{R}^d} |\nabla\varphi|^2(x)dx], \tag{1.3.32}$$

with

$$\tilde{\Phi}_d^\kappa(c) = \{\varphi \in H^1(\mathbb{R}^d) : \int_{\mathbb{R}^d} \varphi^2(x)dx = 1, \int_{\mathbb{R}^d} (1 - e^{-\kappa c\varphi^2(x)})^2 dx \geq 1\}. \tag{1.3.33}$$

This rate function is similar to (1.3.21) except we replace κ_a by κ. To do this, we need to use the scaling relation for the rate function described in (4.1) and (4.2) in [BBH04]. Let $\phi \in H^1(\mathbb{R}^d)$. For $p, q > 0$, define $\varphi \in H^1(\mathbb{R}^d)$ by

$$\phi(x) = q\varphi(x/p). \tag{1.3.34}$$

Then, we have the relations

$$||\nabla\phi||_2^2 = q^2 p^{d-2} ||\nabla\varphi||_2^2, \qquad \int (1 - e^{-\phi^2})^2 = p^d \int (1 - e^{-q^2\varphi^2})^2. \tag{1.3.35}$$

Our aim is to re-write the rate function in (1.3.26) in terms of φ. By setting $q = a^{-1/2}$ and $p = a^{1/d}$, and using the relation in (1.3.34), we get

$$\int_{\mathbb{R}^d} \phi^2(x)dx = \int_{\mathbb{R}^d} \frac{1}{a}\varphi^2(x/a^{1/d})dx = \frac{1}{a}\big(a^{\frac{1}{d}}\big)^d \int_{\mathbb{R}^d} \varphi^2(y)dy = \int_{\mathbb{R}^d} \varphi^2(y)dy.$$

Therefore, the first constraint in (1.3.27) becomes

$$\int_{\mathbb{R}^d} \varphi^2(x)dx = 1. \tag{1.3.36}$$

Moreover, by using the second relation in (1.3.35), the second constraint in (1.3.27) becomes

$$\int_{\mathbb{R}^d} \left(1 - e^{-\kappa\phi^2(x)}\right)^2 dx = a\int_{\mathbb{R}^d} \left(1 - e^{-(\kappa/a)\varphi^2(x)}\right)^2 dx$$

Hence, we get

$$\int_{\mathbb{R}^d} \left(1 - e^{-(\kappa/a)\varphi^2(x)}\right)^2 dx \geq 1. \tag{1.3.37}$$

Combining (1.3.36) and (1.3.37) we get (1.3.33). Finally, by using the first relation in (1.3.35), we have

$$\begin{aligned}\int_{\mathbb{R}^d} |\nabla\phi|^2(x)dx &= (a^{-1/2})^2 \cdot (a^{1/d})^{d-2} \int_{\mathbb{R}^d} |\nabla\phi|^2(x)dx \\ &= a^{\frac{d-2}{d}} \cdot \frac{1}{a} \int_{\mathbb{R}^d} |\nabla\phi|^2(x)dx.\end{aligned}$$

Therefore, the rate function $\hat{I}^{\kappa}_d$ in (1.3.32) follows with $c = 1/a$ and when $a^{(d-2)/d}$ is taken to the main factor in (1.3.28).

1.3.3 The outline

In this section, we give the outline for the proof of Theorem 1.3.7. The full details of the proof will be shown in Chapter 3.

In order to prove Theorem 1.3.7 we will show that the upper bound and the lower bound are the required rate function, i.e.

$$\limsup_{n\to\infty} \frac{1}{n^{\frac{d-2}{d}}} \log \mathbb{P}(J_n \geq an) \leq -\frac{1}{d} L^{\kappa}(a), \tag{1.3.38}$$

$$\liminf_{n\to\infty} \frac{1}{n^{\frac{d-2}{d}}} \log \mathbb{P}(J_n \geq an) \geq -\frac{1}{d} L^{\kappa}(a). \tag{1.3.39}$$

Note that we cannot directly apply the torus technique used in Proposition 1.2.10. This is because *the intersections of ranges may either increase or decrease when wrapped around a torus.* Therefore, we need to use another reflection technique to solve this problem. It turns out that by using this reflection arguments from [BBH04], we are able to apply large deviation results for J_n on a torus. The reflection arguments will be described briefly later. The large deviation result of J_n on a torus will be used for both upper and lower bounds.

The proof in Chapter 3 will be done in the following order: In Section 3.1, we prove the large deviation result for the intersection of ranges on a torus. Most of arguments will be borrowed from Chapter 2. In Section 3.2, We prove (1.3.39) which is done in a similar fashion as in Section 2.2. Finally in Section 3.3, we prove (1.3.38). This will be done by a different technique from a single random walk case. Note that the structure of the proof is identical to that in [BBH04].

Intersection of the ranges of independent random walks on torus

We recall that $\Lambda_N = \big[-\frac{N}{2}, \frac{N}{2}\big)^d$ is the torus of size $N > 0$ with periodic boundary conditions. Let the random walks $(\mathcal{S}_i^1)$ and $(\mathcal{S}_i^2)$ live on $\Lambda_{Nn^{\frac{1}{d}}} \cap \mathbb{Z}^d$ with N fixed. We define

$$\mathcal{J}_n := \sharp\big\{\{\mathcal{S}_j^1\}_{1\leq j\leq n} \cap \{\mathcal{S}_j^2\}_{1\leq j\leq n}\big\} \tag{1.3.40}$$

to be the number of intersection made by the two random walks on $\Lambda_{Nn^{\frac{1}{d}}}$. Our goal for this section is to show that the number of intersection points on torus also satisfies the large deviation principle in the same form as in Proposition 1.2.11.

Proposition 1.3.9. *$\frac{1}{n}\mathcal{J}_n$ satisfies the large deviation principle on $\mathbb{R}_+$ with rate $n^{\frac{d-2}{d}}$*

and with rate function $\frac{1}{d}\hat{L}_N^\kappa$ *where*

$$\hat{L}_N^\kappa(b) = \inf_{\phi\in\hat{\Phi}_N^\kappa(b)} \Big[\int_{\Lambda_N} |\bigtriangledown \phi|^2(x)dx\Big], \tag{1.3.41}$$

where

$$\hat{\Phi}_N^\kappa(b) = \{\phi \in H^1(\Lambda_N) : \int_{\Lambda_N} \phi^2(x)dx = 1, \int_{\Lambda_N} \Big(1 - e^{-\kappa\phi^2(x)}\Big)^2 dx \geq b\}. \tag{1.3.42}$$

Remarks:

1. The rate function of $\frac{1}{n}\mathcal{J}_n$ is different from the one in a single random walk case. In (1.3.42) we have the extra power 2 in the second constraint since $(1-e^{-\kappa\phi^2(x)})^2$ is the density of both random walks visit at site x. Also, the extra factor 2 in the rate function come from our strategy of random walk.

2. Proposition 1.2.11 gives us good control over the $\mathcal{J}_n$.

The global structure of the proof, which we list below, is similar to the proof of Proposition 1.2.11 except that we now consider two random walks. The local structure will be different but only requires minor adaptations.

- Section 3.1.1: Firstly, for $j = 1, 2$ and $\epsilon > 0$, we introduce the *skeleton walk*

$$\mathbb{S}_{n,\epsilon}^j = \{\mathcal{S}^j_{i\epsilon n^{\frac{2}{d}}}\}_{1\leq i\leq \frac{1}{\epsilon}n^{\frac{d-2}{d}}}. \tag{1.3.43}$$

  Then, we will show that the difference between $\frac{1}{n}\mathcal{J}_n$ and $\mathbb{E}_{n,\epsilon}^{(2)}\frac{1}{n}\mathcal{J}_n$ is negligible in the limit as $n \to \infty$ followed by $\epsilon \downarrow 0$, where $\mathbb{E}_{n,\epsilon}^{(2)}$ denotes the conditional expectation given $\mathbb{S}_{n,\epsilon}^j$.

- Section 3.1.2: We represent $\mathbb{E}_{n,\epsilon}^{(2)}\frac{1}{n}\mathcal{J}_n$ as a continuous function of two pair empirical measures

$$L_{n,\epsilon}^j = \epsilon n^{-\frac{d-2}{d}} \sum_{i=1}^{\frac{1}{\epsilon}n^{(d-2)/d}} \delta_{(n^{-1/d}\mathcal{S}^j_{(i-1)\epsilon n^{2/d}},\, n^{-1/d}\mathcal{S}^j_{i\epsilon n^{2/d}})}, \qquad j = 1, 2. \tag{1.3.44}$$

  It turns out that, after a little modification, the continuous function is a product of function of each pair empirical measure. Therefore, we can directly apply the results in Section 2.1.2 to complete the proof.

- Section 3.1.3: We perform the limit $\epsilon \downarrow 0$ to get appropriate approximation for the variational formula in the large deviation principle for $\mathbb{E}_{n,\epsilon}\frac{1}{n}\mathcal{J}_n$. We also derive the large deviation principle of $\frac{1}{n}\mathcal{J}_n$ in this section.

Note that to complete (1.3.38) and (1.3.39), we require an analogous result of Proposition 1.2.13 but for rate function defined in Theorem 1.3.7.

Proposition 1.3.10. $\lim_{N\to\infty} \hat{L}_N^{\kappa}(b) = L^{\kappa}(b)$ *for all* $b > 0$ *where* L^{κ} *is the rate function defined in Theorem 1.3.7.*

Lower bound

To prove (1.3.39) is straightforward as we can use a similar technique as in the proof of the lower bound case of the problem on range in (1.2.29) from Section 2.2. Let $C^2_{Nn^{1/d}}(n)$ to be the event that both random walks will not hit the boundary of $\Lambda_{Nn^{1/d}}$ until time n . Then, we find the lower bound in probability of the event $\{J_n \geq an\}$ by considering the inequality

$$\mathbb{P}(J_n \geq an) \geq \mathbb{P}\big(\mathcal{J}_n \geq an, C^2_{Nn^{1/d}}(n)\big). \tag{1.3.45}$$

Then, we will deduce that the right hand side of (1.3.45) give an appropriate limit for our result, and the limit converges to our required rate function when we expand the size of the torus.

Upper bound

The proof of (1.3.38) follows from the proof of Proposition 4 of [BBH04]. The key idea is to make various random reflections in each direction in such a way that after all the reflections are made, the reflected walks stay inside a very large n-dependent box and the number of intersections neither increases nor decreases. Since the reflected walks stay inside this large box, they behaves similar to random walks on the torus. Hence, we can apply Proposition 1.3.9. The proof will be divided into five steps, and we give the outline here.

1. Section 3.3.1: We introduce the general setup for the proof and proof the preliminary results. There are two main quantities. Define,

$$\Theta_{Nn^{1/d}} := \Big[-\frac{1}{2}Nn^{1/d}, -\frac{1}{2}Nn^{1/d}\Big)^d, \tag{1.3.46}$$

to be a d-dimensional box of side-length $Nn^{1/d}$. Note that, we can partition $\mathbb{Z}^d$ to box of side-length $Nn^{1/d}$ by the following:

$$\mathbb{Z}^d = \bigcup_{z\in\mathbb{Z}^d} \Theta_{Nn^{1/d}}(z), \tag{1.3.47}$$

where $\Theta_{Nn^{1/d}}(z) = \Theta_{Nn^{1/d}} + zNn^{1/d}$. Also, let $Q_{\eta,N,n}$ denote the $\frac{1}{2}\eta n^{1/d}$-

neighborhood of the faces of the boxes, i.e.

$$Q_{\eta,N,n} = \bigcup_{z\in\mathbb{Z}^d} \Big([\Theta_{Nn^{1/d}} \setminus \Theta_{(N-\eta)n^{1/d}}] + zNn^{1/d}\Big). \tag{1.3.48}$$

Assume N/η is an *even* integer. Note that, if we shift $Q_{\eta,N,n}$ by $\eta n^{1/d}$ altogether N/η times in each of the d directions and in every possible combinations we obtain $(N/\eta)^d$ copies of $Q_{\eta,N,n}$. We label this $Q^x_{\eta,N,n}$ for $x = (x_1,\dots,x_d) \in \big\{0,\dots,\frac{N}{\eta}-1\big\}^d$.

2. Section 3.3.2: We start analysing $Q_{\eta,N,n}$. The key part of this section is to show that amongst $Q^x_{\eta,N,n}$, there exists a copy at which random walks behaves "nicely". We call the copy $Q^X_{\eta,N,n}$. Next, we categorise $\Theta_{Nn^{1/d}}(z)$. We denote the box $\Theta_{Nn^{1/d}}(z_1)$ to be a *popular box* if at least one of the two random walks spend a considerable time inside the box, i.e.,

$$\sharp\big\{\{\mathcal{S}^1_j\}_{1\le j\le n} \cap \Theta_{Nn^{1/d}}(z_1)\big\} > \epsilon \text{ or } \sharp\big\{\{\mathcal{S}^2_j\}_{1\le j\le n} \cap \Theta_{Nn^{1/d}}(z_1)\big\} > \epsilon,$$

for $\epsilon > 0$. Then, we define *unpopular boxes* to be those boxes that are not popular boxes. Then, we will do reflection based on the position on the popular boxes. The reflection procedure will be explained in this section and the example of the procedure will be given in Figure 3-2 and Figure 3-3. Then, We end the section by introducing two further important results **(i)** By making the reflection, the reflected walks stay inside a very large n-dependent box, while the cost of making reflection is negligible, and **(ii)** The contribution made at unpopular boxes and the contribution made near some specific area(the boundary) of popular boxes can be neglected.

3. Section 3.3.3 and 3.3.4: We complete the proof of the two results.

4. Section 3.3.5: We complete the proof of (1.3.38) by collecting all the results from the previous sections.

Chapter 2

Large deviation for the range of a single random walk

This chapter is structured as follows: In Section 2.1, we first remind the setup we introduced in Section 1.2.2. Also, we remind the transition densities of Brownian motion and random walk. Then, we define these transition probabilities on a torus and develop a preliminary result. In Section 2.2, we prove the large deviation principle for the range of a random walk on torus. The proof will be divided into four main steps. The structure of this section is as described in Section 1.2.3. Finally, we complete the proof of our main result, Theorem 1.2.10, in Section 2.3. This will be done by deriving the upper and lower bounds as explained in Section 1.2.3.

The content of this chapter is based on Phetpradap [Phe11].

2.1 Transition probability of a random walk

In this section, we remind basic results of Brownian transition kernel and the local central limit theorem. Then, we define these transition probabilities on a torus. We then focus on the local central limit theorem for a random walk on a torus and develop a result on the transition probability of the random walk on torus. The result will be used to acquire the large deviation principle on the range of the random walk on a torus in Section 2.2.2.

Firstly, we recall the compactification we described at the beginning of Section 1.2.3: For $N \in \mathbb{N}$ even, let Λ_N be the torus of size $N > 0, \Lambda_N = [-\frac{N}{2}, \frac{N}{2})^d$ with periodic boundary conditions. To prove Proposition 1.2.11 in Section 2.2, we will let the random walk live on $\Lambda_{Nn^{\frac{1}{d}}} \cap \mathbb{Z}^d$ with N fixed. We denote by $(\mathcal{S}_n : n = 1, 2, \ldots)$ the corresponding random walk on $\Lambda_{Nn^{\frac{1}{d}}} \cap \mathbb{Z}^d$ and by $\mathcal{R}_n$ the number of lattice sites visited by the random walk on the torus up to time n.

Due to the discreteness property of random walk, we need to take into account the parity issue of random walk. Let $x = (x_1, \dots, x_d)$ and $y = (y_1, \dots, y_d)$ with $x, y \in \mathbb{Z}^d$. We say that:

- The site x is in even(odd) parity if the sum $x_1 + \dots + x_d$ is even(odd).
- The site x and time n have the same parity if $x_1 + \dots + x_d + n$ is even.
- The sites x and y have the same parity if $x_1 + \dots + x_d + y_1 + \dots + y_d$ is even.
- The sites x, y and time n have the same parity if $x_1 + \dots + x_d + y_1 + ... + y_d + n$ is even.

We first denote the transition probability of random walk from x to y at time n by

$$\mathfrak{p}_n(x, y) := \mathbb{P}(S_n = y | S_0 = x). \tag{2.1.1}$$

We also denote $\mathfrak{p}_n(x) = \mathfrak{p}_n(0, x)$. Note that by *the local central limit theorem* (see Lemma 17.6 [Rév05]), assume x, y and n have the same parity, we get

$$\mathfrak{p}_n(x, y) = 2\left(\frac{d}{2\pi n}\right)^{d/2} \exp\left[-\frac{d|y-x|^2}{2n}\right] + A_n(x, y), \tag{2.1.2}$$

where

$$A_n(x, y) = \min\left(O(n^{-(d+2)/2}), O(|y-x|^{-2} n^{-d/2})\right). \tag{2.1.3}$$

And, trivially, if x, y and n do not have the same parity, then $\mathfrak{p}_n(x, y) = 0$. We quote two further results on the properties of the random walk on $\mathbb{Z}^d$ from (2.1.2) and Lemma 17.8 [Rév05].

Lemma 2.1.1. *Let $d \geq 3$. Assume x and n have the same parity.*

(a) Assume that x is in even parity. Then,

$$\mathbb{P}(S_{2n} = x) \begin{cases} = O(n^{-d/2}), & \text{if } n > |x|^2, \\ \leq O\left(n^{-d/2} \exp\left[-\frac{|x|^2}{2n}\right]\right), & \text{if } n < |x|^2. \end{cases}$$

(b) As $|x| \to \infty$, There exists a positive constant C_d such that

$$\mathbb{P}(S_n = x \text{ for some } n) = \frac{C_d + o(1)}{|x|^{d-2}}.$$

Lemma 2.1.1(a) shows the probability that a random walk makes a visit at site x at time $2n$ is bounded and depends on the position of the site, while Lemma 2.1.1 (b) shows the probability that random walk will ever visit site x. Next, we define the transition probability for the random walk on the torus $\Lambda_{Nn^{1/d}}$. We denote the probability by:

$$\mathfrak{p}_n^{\pi}(x,y) := \mathbb{P}(\mathcal{S}_n = y | \mathcal{S}_0 = x). \tag{2.1.4}$$

Using the periodicity of $\Lambda_{Nn^{1/d}}$, we can deduce that

$$\mathfrak{p}_n^{\pi}(x,y) = \sum_{z\in\mathbb{Z}^d} \mathfrak{p}_n(x, y + zNn^{1/d}). \tag{2.1.5}$$

We also denote $\mathfrak{p}_n^{\pi}(x) = \mathfrak{p}_n^{\pi}(0,x)$.

Next, we denote the transition probability of Brownian motion on $\mathbb{R}^d$. For $x, y \in \mathbb{R}^d$, define $p_t(x,y)$ to be the *Brownian transition kernel* from the point x to the point y at time t, i.e.

$$p_t(x,y) = \left(\frac{1}{2\pi t}\right)^{d/2} \exp\left[-\frac{|y-x|^2}{2t}\right]. \tag{2.1.6}$$

We also denote $p_t(x) = p_t(0,x)$.

It is worth pointing out that for Brownian motion, the transition kernel is Gaussian, while for random walk the transition density is not Gaussian, but its limit converges to Gaussian.

We end the section by showing that the transition probability for a random walk on torus is bounded by a multiple of the transition probability for a random walk on $\mathbb{Z}^d$. In order to apply this result with our proof directly, we consider the transition probability of random walk on $\Lambda_{Nn^{1/d}}$ from point $an^{1/d}$ to point $bn^{1/d}$ at time $\epsilon n^{2/d}$ (We may assume for now that $an^{1/d}, bn^{1/d}$ are on $\Lambda_{Nn^{1/d}} \cap \mathbb{Z}^d$ and $\epsilon n^{2/d}$ is an integer. Also, assume that $an^{1/d}, bn^{1/d}$ and $\epsilon n^{2/d}$ have the same parity). The choice of timelength is directly from the length of skeleton introduced in (1.2.33). The reason will become clearer in Section 2.2.1. Therefore, by (2.1.4) and (2.1.5), we get

$$\mathfrak{p}^{\pi}_{\epsilon n^{2/d}}(an^{1/d}, bn^{1/d}) = \sum_{z\in\mathbb{Z}^d} \mathfrak{p}_{\epsilon n^{2/d}}\big(an^{1/d}, (b+zN)n^{1/d}\big), \tag{2.1.7}$$

where

$$\mathfrak{p}_{\epsilon n^{2/d}}(an^{1/d}, bn^{1/d}) = 2\left(\frac{d}{2\pi\epsilon n^{2/d}}\right)^{d/2} \exp\left[-\frac{d|bn^{1/d} - an^{1/d}|^2}{2\epsilon n^{2/d}}\right] + \tilde{A}_n(a,b), \tag{2.1.8}$$

at which $\tilde{A}_n(a,b) = \min\Big(O\big(n^{-1-\frac{2}{d}}\big), O\big(|b-a|^{-2}n^{-1-\frac{2}{d}}\big)\Big)$. Note that the O-term may depend on ϵ, but not any other variables.

Next we start to determine the transition probability in (2.1.7). Let $bn^{1/d}$ be a point on $\Lambda_{Nn^{1/d}}$. We can see that any points in the form $(b+zN)n^{1/d}$ for $z \in \mathbb{Z}^d$ will project to the point b on Λ_N. We define $b^*n^{1/d}$ the point amongst this form that provides the shortest distance from $an^{1/d}$. Since we assume that N is even and a and b have the same parity, it is clear that

$$\mathfrak{p}_m(an^{1/d}, b^*n^{1/d}) = \sup_{z\in\mathbb{Z}^d} \big\{\mathfrak{p}_m\big(an^{1/d}, (b+zN)n^{1/d}\big)\big\}. \tag{2.1.9}$$

Finally, note that if we unwrap the torus $\Lambda_{Nn^{1/d}}$, it will look like drawing boxes on $\mathbb{Z}^d$ where each box has sidelength $Nn^{1/d}$ and centred at points $zNn^{1/d}$ for $z \in \mathbb{Z}^d$. We will call the box with centre 0 the central box. Also, we will call the boxes adjacent to the central box (that is the box centred at $zNn^{1/d}$ with $||z||_\infty = 1$) the first shell, and we will call the boxed adjacent to the first shell the second shell and so on. Without loss on generality, we always put the point $an^{1/d}$ on the central box. It can be seen that $b^*n^{1/d}$ must lie either in the central box or on the first shell.

Lemma 2.1.2. *For $a, b \in \Lambda_N$ with O-term independent of a and b,*

$$\mathfrak{p}^{\pi}_{\epsilon n^{2/d}}(an^{1/d}, bn^{1/d}) \leq 3^d \mathfrak{p}_{\epsilon n^{2/d}}(an^{1/d}, b^*n^{1/d}) + O(n^{-1}).$$

Proof. The number of boxes in the first shell, including the central box, is 3^d. Since $b^*n^{1/d}$ lies either on the central box or the first shell, we can conclude by (2.1.9) that

$$\sum_{z\in\mathbb{Z}^d:||z||_\infty\leq 1} \mathfrak{p}_{\epsilon n^{2/d}}(an^{1/d}, (b+zN)n^{1/d}) \leq 3^d \mathfrak{p}_{\epsilon n^{2/d}}(an^{1/d}, b^*n^{1/d}). \tag{2.1.10}$$

Now, for $k \geq 2$, if the point $bn^{1/d}$ lies on the box on the k^{th} shell, then $|an^{1/d} - bn^{1/d}| > (k-1)Nn^{1/d}$. The number of boxes on each k^{th} shell is $(2k+1)^d - (2k-1)^d$ and hence,

$$\begin{aligned}\sum_{z\in\mathbb{Z}^d:||z||_\infty\geq 2} \mathfrak{p}_{\epsilon n^{2/d}}\big(an^{1/d}, (b+zN)n^{1/d}\big) &\leq \sum_{k=1}^{\infty}\Big((2k+1)^d - (2k-1)^d\Big)\mathfrak{p}_{\epsilon n^{2/d}}(0, \mathbf{k}Nn^{1/d}) \\ &\leq \sum_{k=1}^{\infty}(3k)^d \mathfrak{p}_{\epsilon n^{2/d}}\big(0, \mathbf{k}Nn^{1/d}\big), \end{aligned} \tag{2.1.11}$$

where $\mathbf{k} = (k, 0, \ldots, 0)$ is a d-dimensional vector with value k in dimension 1 and zero in the other dimensions. From Lemma 2.1.1 (a) we get,

$$
\begin{aligned}
\mathfrak{p}_{\epsilon n^{2/d}}(0, \mathbf{k}Nn^{1/d}) &\le O\bigg((\frac{\epsilon n^{2/d}}{2})^{-d/2} \exp\Big[- \frac{|\mathbf{k}Nn^{1/d}|^2}{2\epsilon n^{2/d}} \Big] \bigg) \\
&= \Big(\frac{\epsilon}{2}\Big)^{-d/2} \exp\Big[- \frac{k^2 N^2}{\epsilon} \Big] O(n^{-1}),
\end{aligned}
\tag{2.1.12}
$$

subject to the condition $\epsilon < (kN)^2$. Substitute (2.1.12) into (2.1.11) we get,

$$
\sum_{z \in \mathbb{Z}^d : ||z||_\infty \ge 2} \mathfrak{p}_{\epsilon n^{2/d}}\big(an^{1/d}, (b + zN)n^{1/d}\big) = O(n^{-1}). \tag{2.1.13}
$$

Combining (2.1.10) and (2.1.13), we get the Lemma. $\square$

2.2 Random walk on a torus

In this section, we complete the proof of Proposition 1.2.11. We recall and rename the proposition for the ease of reading.

Proposition 2.2.1. *Let $d \ge 3$, then $\frac{1}{n}\mathcal{R}_n$ satisfies a LDP on $\mathbb{R}^+$ with speed $n^{\frac{d-2}{d}}$ and rate function $\frac{1}{d}J_N^\kappa$, where*

$$
J_N^\kappa(b) = \inf_{\phi \in \partial\Phi_N^\kappa(b)} [\frac{1}{2} \int_{\Lambda_N} |\nabla \phi|^2(x) dx] \tag{2.2.1}
$$

with

$$
\partial\Phi_N^\kappa(b) = \{\phi \in H^1(\Lambda_N) : \int_{\Lambda_N} \phi^2(x) dx = 1, \int_{\Lambda_N} \big(1 - e^{-\kappa\phi^2(x)}\big) dx = b\}. \tag{2.2.2}
$$

For any functions f_n and g_n , we write $f_n(x) \approx g_n(x)$ if and only if,

$$
\lim_{n \to \infty} \frac{\log f_n(x)}{\log g_n(x)} = 1. \tag{2.2.3}
$$

2.2.1 Approximation of $\frac{1}{n}\mathcal{R}_n$ by $\mathbb{E}_{n,\epsilon}\frac{1}{n}\mathcal{R}_n$

We recall and rename the skeleton walk defined in (1.2.33).

$$
\mathbb{S}_{n,\epsilon} = \{\mathcal{S}_{i\epsilon n^{\frac{2}{d}}}\}_{1 \le i \le \frac{1}{\epsilon} n^{\frac{d-2}{d}}}. \tag{2.2.4}
$$

Also, recall that $\mathbb{E}_{n,\epsilon}$ denotes the conditional expectation given $\mathbb{S}_{n,\epsilon}$ and $\mathbb{P}_{n,\epsilon}$ denote the conditional probability given $\mathbb{S}_{n,\epsilon}$. In this section, we show that $\mathcal{R}_n$ can be approximated by its conditional expectation given $\mathbb{S}_{n,\epsilon}$:

Proposition 2.2.2. *For all $\delta > 0$,*

$$\lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}\frac{1}{n^{\frac{d-2}{d}}}\log\mathbb{P}\big(\tfrac{1}{n}|\mathcal{R}_n-\mathbb{E}_{n,\epsilon}\mathcal{R}_n|\geq\delta\big)=-\infty.$$

Proof. **(1)** For $1\leq i\leq\frac{1}{\epsilon}n^{\frac{d-2}{d}}$, let

$$\mathcal{W}_i=\big\{\mathcal{S}_j:(i-1)\epsilon n^{\frac{2}{d}}\leq j\leq i\epsilon n^{\frac{2}{d}}\big\}. \tag{2.2.5}$$

Then, it is easy to see that

$$\frac{1}{n}\mathcal{R}_n=\frac{1}{n}\sharp\Bigg\{\bigcup_{i=1}^{\frac{1}{\epsilon}n^{\frac{d-2}{d}}}\mathcal{W}_i\Bigg\}. \tag{2.2.6}$$

Now, for $K>0$, let:

$$\mathcal{J}_{n,\epsilon}^K=\big\{1\leq i\leq\tfrac{1}{\epsilon}n^{\frac{d-2}{d}}:\tfrac{1}{n^{1/d}}|\mathcal{S}_{(i-1)\epsilon n^{\frac{2}{d}}}-\mathcal{S}_{i\epsilon n^{\frac{2}{d}}}|\leq K\sqrt{\epsilon}\big\}. \tag{2.2.7}$$

and define,

$$\frac{1}{n}\mathcal{R}_{n,\epsilon}^K=\frac{1}{n}\sharp\Big\{\bigcup_{i\in\mathcal{J}_{n,\epsilon}^K}\mathcal{W}_i\Big\}, \tag{2.2.8}$$

$$\frac{1}{n}\hat{\mathcal{R}}_{n,\epsilon}^K=\frac{1}{n}\sharp\Big\{\bigcup_{i\notin\mathcal{J}_{n,\epsilon}^K}\mathcal{W}_i\Big\}. \tag{2.2.9}$$

Since $0\leq\frac{1}{n}\mathcal{R}_n-\frac{1}{n}\mathcal{R}_{n,\epsilon}^K\leq\frac{1}{n}\hat{\mathcal{R}}_{n,\epsilon}^K$, we have

$$\frac{1}{n}|\mathcal{R}_n-\mathbb{E}_{n,\epsilon}\mathcal{R}_n|\leq\frac{1}{n}|\mathcal{R}_{n,\epsilon}^K-\mathbb{E}_{n,\epsilon}\mathcal{R}_{n,\epsilon}^K|+\frac{1}{n}\hat{\mathcal{R}}_{n,\epsilon}^K+\frac{1}{n}\mathbb{E}_{n,\epsilon}\hat{\mathcal{R}}_{n,\epsilon}^K. \tag{2.2.10}$$

Therefore, to prove Proposition 2.2.2, we need to prove the two following results:

$$\lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}\frac{1}{n^{\frac{d-2}{d}}}\log\mathbb{P}\Big(\frac{1}{n}|\mathcal{R}_{n,\epsilon}^K-\mathbb{E}_{n,\epsilon}\mathcal{R}_{n,\epsilon}^K|\geq\delta\Big)=-\infty\quad\text{for all}\quad 0<\delta<1,K>K_0(\delta). \tag{2.2.11}$$

$$\lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}\frac{1}{n^{\frac{d-2}{d}}}\log\mathbb{P}\Big(\frac{1}{n}\hat{\mathcal{R}}_{n,\epsilon}^K\geq\delta\Big)=-\infty\quad\text{for all}\quad 0<\delta<1,K>K_0(\delta). \tag{2.2.12}$$

We can ignore the third term of the right hand side of (2.2.10). Note that, $\frac{1}{n}\hat{\mathcal{R}}_{n,\epsilon}^K\leq$

$\frac{1}{n}|\Lambda_{Nn^{\frac{1}{d}}} \cap \mathbb{Z}^d| = N^d$ and hence $\mathbb{E}_{n,\epsilon}(\frac{1}{n}\hat{\mathcal{R}}^K_{n,\epsilon}) \le \frac{\delta}{2} + N^d \mathbb{P}_{n,\epsilon}(\frac{1}{n}\hat{\mathcal{R}}^K_{n,\epsilon} \ge \frac{\delta}{2})$. By using Markov's inequality,

$$\mathbb{P}\Big(\mathbb{E}_{n,\epsilon}\Big(\frac{1}{n}\hat{\mathcal{R}}^K_{n,\epsilon}\Big) \ge \delta\Big) \le \mathbb{P}\Big(\mathbb{P}_{n,\epsilon}\Big(\frac{1}{n}\hat{\mathcal{R}}^K_{n,\epsilon} \ge \frac{\delta}{2}\Big) \ge \frac{\delta}{2N^d}\Big) \le \frac{2N^d}{\delta}\mathbb{P}\Big(\frac{1}{n}\hat{\mathcal{R}}^K_{n,\epsilon} \ge \frac{\delta}{2}\Big),$$

which will be studied in (2.2.12).

(2) To prove (2.2.12), we estimate by Markov's inequality that

$$\begin{aligned}
&\mathbb{P}(\frac{1}{n}\hat{\mathcal{R}}^K_{n,\epsilon} \ge \delta) \\
&\quad \le \exp\Big[-\frac{\delta n^{(d-2)/d}}{2\epsilon}\Big]\mathbb{E}\Big(\exp\Big[\frac{n^{\frac{d-2}{d}}}{2\epsilon}\sum_{i=1}^{\frac{1}{\epsilon}n^{\frac{d-2}{d}}}\frac{1}{n}\sharp\{\mathcal{W}_i\}\mathbf{1}\{i \notin \mathcal{J}^K_{n,\epsilon}\}\Big]\Big) \\
&\quad = \exp\Big[-\frac{\delta n^{(d-2)/d}}{2\epsilon}\Big]\mathbb{E}\Big(\exp\Big[\frac{n^{\frac{d-2}{d}}}{2\epsilon}\frac{1}{n}\sharp\{\mathcal{W}_1\}\mathbf{1}\{1 \notin \mathcal{J}^K_{n,\epsilon}\}\Big]\Big)^{\frac{1}{\epsilon}n^{\frac{d-2}{d}}} \\
&\quad = \exp\Big[-\frac{\delta n^{(d-2)/d}}{2\epsilon}\Big]\Big\{1 + \mathbb{E}\Big(\mathbf{1}\{1 \notin \mathcal{J}^K_{n,\epsilon}\}\Big(\exp\Big[\frac{1}{2\epsilon n^{2/d}}\sharp\{\mathcal{W}_1\}\Big] - 1\Big)\Big)\Big\}^{\frac{1}{\epsilon}n^{\frac{d-2}{d}}} \\
&\quad \le \exp\Big[-\frac{\delta n^{(d-2)/d}}{2\epsilon}\Big]\Big\{1 + \sqrt{\delta_K \mathbb{E}\Big(\exp\Big[\frac{1}{\epsilon n^{2/d}}\sharp\{\mathcal{W}_1\}\Big]\Big)}\Big\}^{\frac{1}{\epsilon}n^{\frac{d-2}{d}}}, \qquad (2.2.13)
\end{aligned}$$

where $\delta_K = \sup_{n\ge 1}\mathbb{P}\big(\frac{1}{n^{1/d}}|\mathcal{S}_{\epsilon n^{2/d}}| > K\sqrt{\epsilon}\big)$ and we use the Cauchy-Schwarz inequality in the last step. However,

$$\mathbb{E}\Big(\exp\Big[\frac{1}{\epsilon n^{2/d}}\sharp\{\mathcal{W}_1\}\Big]\Big) = \mathbb{E}\Big(\exp\Big[\frac{1}{\epsilon n^{2/d}}\mathcal{R}_{\epsilon n^{2/d}}\Big]\Big) \le \mathbb{E}\Big(\exp\Big[\frac{1}{\epsilon n^{2/d}}\epsilon n^{2/d}\Big]\Big) = e.$$

Hence, from (2.2.13), for all $\epsilon, K > 0$

$$\limsup_{n\to\infty}\frac{1}{n^{\frac{d-2}{d}}}\log\mathbb{P}\big(\tfrac{1}{n}\hat{\mathcal{R}}^K_{n,\epsilon} \ge \delta\big) \le \frac{-\delta}{2\epsilon} + \frac{1}{\epsilon}\sqrt{e\delta_K}.$$

Note that, $\lim_{K\to\infty}\delta_K = 0$. Therefore, there exists $K_0(\delta)$ such that $\sqrt{e\delta_K} \le \frac{\delta}{4}$ for $K \ge K_0(\delta)$. For such K, we now let $\epsilon \downarrow 0$ and (2.2.12) follows.

(3) Before we we prove (2.2.11), we first refer a result due to Talagrand (Theorem 2.4.1, [Tal95]). For $i \in \mathbb{N}$, consider a probability space $(\Omega_i, \Sigma_i, \mu_i)$, and the M-fold product

$$\Big(\prod_{i=1}^M \Omega_i, \bigotimes_{i=1}^M \Sigma_i, \bigotimes_{i=1}^M \mu_i\Big)$$

for $M \in \mathbb{N}$ fixed. We denote P the product probability $\bigotimes_{i=1}^{M} \mu_i$. We use the convention $(x_1, \ldots, x_m) =: x \in \prod_{i=1}^{M} \Omega_i$ for a point in the product space.

Theorem 2.2.3. *Let $A \subset \prod_{i=1}^{M} \Omega_i$ be measurable and $t > 0$ such that, for $i = 1, \ldots, M$, $\int\int_{\Omega_i^2} \exp[t \cdot h_i(z, z')] d\mu_i(z)\mu_i(z') < \infty$, where h_i is a measurable function. Set, $\nu_i(\omega, \omega') = \max(h_i(\omega, \omega'), h_i(\omega', \omega))$ for $\omega, \omega' \in \Omega_i$. Then,*

$$\int_{\Omega_1 \times \ldots \times \Omega_M} \exp\left[t f_h(A, x)\right] dP(x) \leq \frac{1}{P(A)} \prod_{i=1}^{M} \left(\int_{\Omega_i^2} \cosh\left(t \cdot \nu_i(\omega, \omega')\right) d\mu_i(\omega) d\mu_i(\omega') \right),$$

where

$$f_h(A, x) = \inf \Big\{ \sum_{i \leq M} h_i(x_i, y_i); y \in A \Big\}.$$

Remark: Note that Theorem 2.4.1 in [Tal95] gives a result when all $\Omega_i, i = 1, \ldots, M$ are identical. However, it is straightforward from the arguments of the proof of his theorem that the analogous result also holds for a M-fold product since the proof is done by induction over M. This extension is also suggested in Remark 2.1.3 of that paper.

To prove (2.2.11), we do the following: Conditionally on $\mathbb{S}_{n,\epsilon}$, the $\mathcal{W}_i$ are independent random subsets of $\Lambda_{Nn^{\frac{1}{d}}}$. Let $\mathcal{T}$ be the set of subsets of $\Lambda_{Nn^{\frac{1}{d}}} \cap \mathbb{Z}^d$. The mapping $d : \mathcal{T} \times \mathcal{T} \to [0, \infty)$ with

$$d(A, B) = \tfrac{1}{n} \sharp\{A \triangle B\} = \tfrac{1}{n} \sharp\{(A \setminus B) \cup (B \setminus A)\} \tag{2.2.14}$$

defines a metric on $\mathcal{T}$. Then, $\mathbb{P}_{n,\epsilon}$ defines a product measure on $\prod_{i=1}^{\frac{1}{\epsilon} n^{(d-2)/d}} \mathcal{T}$, which we denote by the same symbol $\mathbb{P}_{n,\epsilon}$. Define,

$$M(C) = \sharp\Big\{ \bigcup_{i \in \mathcal{J}_{n,\epsilon}^K} C_i \Big\}, \qquad \Big(C = \{C_i\} \in \prod_{i=1}^{\frac{1}{\epsilon} n^{(d-2)/d}} \mathcal{T}\Big).$$

Note that conditionally on $\mathbb{S}_{n,\epsilon}$ fixes, $\mathcal{J}_{n,\epsilon}^K$ and M are Lipschitz in the sense that

$$\big| M(C) - M(C') \big| \leq \sum_{i \in \mathcal{J}_{n,\epsilon}^K} \tfrac{1}{n} \sharp\{C_i \triangle C_i'\}, \qquad \Big(C, C' \in \prod_{i=1}^{\frac{1}{\epsilon} n^{(d-2)/d}} \mathcal{T}\Big). \tag{2.2.15}$$

Now, denote by $m_{n,\epsilon}^K$ the median of the distribution of $Z := \sharp\{ \bigcup_{i \in \mathcal{J}_{n,\varepsilon}^K} \mathcal{W}_i \} = \mathcal{R}_{n,\epsilon}^K$ under

$\mathbb{P}_{n,\epsilon}$, i.e. $m^K_{n,\epsilon} = \inf\{m : \mathbb{P}_{n,\epsilon}(Z < m) \geq \frac{1}{2}\}$, and define

$$A = \{C \in \prod_{i=1}^{\frac{1}{\epsilon}n^{(d-2)/d}} \mathcal{T} : M(C) \leq m^K_{n,\epsilon}\}.$$

Note that for $\epsilon \in (0,1]$ fixed and n large enough, there exists constant $0 < \xi < \frac{1}{2}$ such that $\xi < \mathbb{P}_{n,\epsilon}(A) < 1-\xi$. This is because of the size of the atomic masses of the distribution of Z is bounded away from 1 in $\epsilon \in (0,1]$ and $n \in \mathbb{N}$.

We applying Theorem 2.2.3 with the mapping d defined in (2.2.14) as the function $h, t = \lambda/n, x = \{W_i\}$ for the conditional expectation $\mathbb{E}_{n,\epsilon}$. Hence, we get

$$\mathbb{E}_{n,\epsilon}\Big(\exp\Big[\lambda f(A, \{\mathcal{W}_i\})\Big]\Big) \leq \xi^{-1} \prod_{i \in \mathcal{J}^K_{n,\epsilon}} \mathbb{E}_{n,\epsilon}\Big(\cosh\Big[\frac{\lambda}{n}\sharp\{\mathcal{W}_i \triangle \mathcal{W}'_i\}\Big]\Big),$$

where

$$f\big(A, \{C_i\}\big) = \inf_{C' \in A} \sum_{i \in \mathcal{J}^K_{n,\varepsilon}} \tfrac{1}{n}\sharp\{C_i \triangle C'_i\},$$

and $\{\mathcal{W}'_i\}$ is an independent copy of $\{\mathcal{W}_i\}$. By Chebyshev's inequality we get

$$\begin{aligned}\mathbb{P}_{n,\epsilon}\big(f(A, \{\mathcal{W}_i\}) \geq \delta\big) &\leq \inf_{\lambda>0} e^{-\lambda\delta}\mathbb{E}_{n,\epsilon}e^{\lambda f\big(A, \{\mathcal{W}_i\}\big)} \\ &\leq \xi^{-1} \inf_{\lambda>0} e^{-\lambda\delta} \prod_{i \in \mathcal{J}^K_{n,\epsilon}} \mathbb{E}_{n,\epsilon}\Big(\cosh\Big[\frac{\lambda}{n}\sharp\{\mathcal{W}_i \triangle \mathcal{W}'_i\}\Big]\Big) =: \Xi^K_{n,\epsilon}(\delta).\end{aligned} \tag{2.2.16}$$

Arguing similarly with $\hat{A} = \big\{C \in \mathcal{T}^{\frac{1}{\epsilon}n^{\frac{d-2}{d}}} : M(C) \geq m^K_{n,\epsilon}\big\}$ and noting that $\mathcal{R}^K_{n,\epsilon} = M\big(\{\mathcal{W}_i\}\big)$, by using (2.2.15), we get

$$\begin{aligned}\mathbb{P}_{n,\epsilon}\big(\tfrac{1}{n}|\mathcal{R}^K_{n,\epsilon} - m^K_{n,\epsilon}| \geq \delta\big) &\leq \mathbb{P}_{n,\epsilon}\big(f(A, \{\mathcal{W}_i\}) \geq \delta\big) + \mathbb{P}_{n,\epsilon}\big(f(\hat{A}, \{\mathcal{W}_i\}) \geq \delta\big) \\ &\leq 2\Xi^K_{n,\epsilon}(\delta).\end{aligned} \tag{2.2.17}$$

(4) Note that,

$$\frac{1}{n}\big|\mathbb{E}_{n,\epsilon}\mathcal{R}^K_{n,\epsilon} - m^K_{n,\epsilon}\big| \leq \frac{\delta}{3} + \frac{1}{n}\sharp\{\Lambda_{Nn^{\frac{1}{d}}} \cap \mathbb{Z}^d\}\mathbb{P}_{n,\epsilon}\Big(\frac{1}{n}\big|\mathcal{R}^K_{n,\epsilon} - m^K_{n,\epsilon}\big| \geq \frac{\delta}{3}\Big), \tag{2.2.18}$$

consequently, since $\frac{1}{n}\mathcal{R}^K_{n,\epsilon} \le N^d$, by (2.2.17) and (2.2.18),

$$\begin{aligned}
&\mathbb{P}_{n,\epsilon}\big(\tfrac{1}{n}|\mathcal{R}^K_{n,\epsilon} - \mathbb{E}_{n,\epsilon}\mathcal{R}^K_{n,\epsilon}| \ge \delta\big) \\
&\qquad\qquad \le \mathbb{P}_{n,\epsilon}\Big(\tfrac{1}{n}|\mathcal{R}^K_{n,\epsilon} - m^K_{n,\epsilon}| \ge \frac{\delta}{3}\Big) + \mathbf{1}\Big(\tfrac{1}{n}|\mathbb{E}_{n,\epsilon}\mathcal{R}^K_{n,\epsilon} - m^K_{n,\epsilon}| \ge \frac{2\delta}{3}\Big) \\
&\qquad\qquad \le 2\Xi^K_{n,\epsilon}\Big(\frac{\delta}{3}\Big) + \mathbf{1}\Big\{\mathbb{P}_{n,\epsilon}\Big(\tfrac{1}{n}|\mathcal{R}^K_{n,\epsilon} - m^K_{n,\epsilon}| \ge \frac{\delta}{3}\Big) \ge \frac{\delta}{3N^d}\Big\} \\
&\qquad\qquad \le 2\Xi^K_{n,\epsilon}\Big(\frac{\delta}{3}\Big) + \mathbf{1}\Big\{2\Xi^K_{n,\epsilon}\Big(\frac{\delta}{3}\Big) \ge \frac{\delta}{3N^d}\Big\}. \qquad (2.2.19)
\end{aligned}$$

By Chebyshev's inequality and (2.2.19) we get, after averaging over $\mathbb{S}_{n,\epsilon}$,

$$\mathbb{P}\big(\tfrac{1}{n}\big|\mathcal{R}^K_{n,\epsilon} - \mathbb{E}_{n,\epsilon}\mathcal{R}^K_{n,\epsilon}\big| \ge \delta\big) \le \Big(1 + \frac{3N^d}{\delta}\Big)\mathbb{E}\Big(2\Xi^K_{n,\epsilon}\Big(\frac{\delta}{3}\Big)\Big). \qquad (2.2.20)$$

Therefore, to prove (2.2.11), it suffices to show that

$$\lim_{\epsilon\downarrow 0}\limsup_{n\to\infty} \frac{1}{n^{\frac{d-2}{d}}} \log \mathbb{E}\big(\Xi^K_{n,\epsilon}(\delta)\big) = -\infty \qquad \forall\ 0<\delta<1, K > K_0(\delta), \qquad (2.2.21)$$

which will follow if we can show a stronger version, namely,

$$\lim_{\epsilon\downarrow 0}\limsup_{n\to\infty} \frac{1}{n^{\frac{d-2}{d}}} \log ||\Xi^K_{n,\epsilon}(\delta)||_\infty = -\infty \qquad \forall\ 0<\delta<1, K > K_0(\delta), \qquad (2.2.22)$$

where $||X||_\infty$ is the infinity norm for a random variable X.

(5) To estimate $\mathbb{E}_{n,\epsilon}\big(\cosh\big[\frac{\lambda\sharp\{\mathcal{W}_i\triangle\mathcal{W}'_i\}}{n}\big]\big)$ from (2.2.16), we pick $\lambda = \frac{c}{\epsilon}n^{\frac{d-2}{d}}$ for $0<c<1$ and use the fact that $\cosh(cd) \le 1 + c^2\exp(d)$ for $0 < c \le 1$ (this can be checked by expanding exponential terms from both sides). Hence,

$$\begin{aligned}
\mathbb{E}_{n,\epsilon}\Big(\cosh\Big[\frac{cn^{\frac{d-2}{d}}\sharp\{\mathcal{W}_i\triangle\mathcal{W}'_i\}}{\epsilon n}\Big]\Big) &\le 1 + c^2\mathbb{E}_{n,\epsilon}\Big(\exp\Big[\frac{\sharp\{\mathcal{W}_i\triangle\mathcal{W}'_i\}}{\epsilon n^{2/d}}\Big]\Big) \\
&\le 1 + c^2\mathbb{E}_{n,\epsilon}\Big(\exp\Big[\frac{\sharp\{\mathcal{W}_i\}}{\epsilon n^{2/d}}\Big]\Big)^2 \\
&\le 1 + c^2e^2, \qquad (2.2.23)
\end{aligned}$$

where in the last inequality, we use $\sharp\{\mathcal{W}_i\} = \mathcal{R}_{\epsilon n^{2/d}} \le \epsilon n^{2/d}$. Therefore, by (2.2.16) and (2.2.23),

$$\begin{aligned}
\Xi^K_{n,\epsilon}(\delta) &\le \xi^{-1}\exp\Big[-\frac{c\delta}{\epsilon}n^{\frac{d-2}{d}}\Big]\prod_{i\in J^K_{n,\epsilon}}\mathbb{E}_{n,\epsilon}\Big(\cosh\Big[\frac{\lambda}{n}|\mathcal{W}_i\triangle\mathcal{W}_i'|\Big]\Big)\\
&\le \xi^{-1}\exp\Big[-\frac{c\delta}{\epsilon}n^{\frac{d-2}{d}}\Big]\prod_{i=1}^{\frac{1}{\epsilon}n^{\frac{d-2}{d}}}(1+c^2e^2)\\
&\le \xi^{-1}\exp\Big[-\frac{c\delta}{\epsilon}n^{\frac{d-2}{d}}\Big]\Big(\exp[c^2e^2]\Big)^{\frac{1}{\epsilon}n^{\frac{d-2}{d}}}\\
&= \xi^{-1}\exp\Big[(-c\delta+c^2e^2)\frac{1}{\epsilon}n^{\frac{d-2}{d}}\Big].
\end{aligned}$$

Now, for any $\delta > 0$, we pick c satisfying $0 < c < \min(1, \delta/e^2)$. Let $n \to \infty$ followed by $\epsilon \downarrow 0$. We then get (2.2.22) and the proof of Proposition 2.2.2 is now completed. □

2.2.2 The LDP for $\mathbb{E}_{n,\epsilon}\frac{1}{n}\mathcal{R}_n$

In this section, we prove the large deviation principle for the conditional expectation of random walk on $\Lambda_{Nn^{1/d}}$ given $\mathbb{S}_{n,\epsilon}$.

We would like to remove n-dependence for the random walk on the torus. In order to do that, we will do a scaling of the torus from the original size of $Nn^{1/d}$ to the size N. The mesh of the random walk on the scaled torus, Λ_N, will now be $n^{-1/d}$ and therefore a point a in the scaled torus corresponds to the point $an^{1/d}$ in the original torus. We will use $\tilde{\mathcal{S}}_n$ for the corresponding position of the random walk on Λ_N and P for its law. However, it may be the case that the point $a = (a_1, \dots, a_d) \in \Lambda_N$ is not on the scaled grid $\Lambda_N \cap n^{-1/d}\mathbb{Z}^d$. Therefore, from now on, we use the following convention: unless stated otherwise, $\tilde{\mathcal{S}}_n = a$ will have the same meaning as $\tilde{\mathcal{S}}_n = \lfloor a\rfloor$ where $\lfloor a\rfloor = (\lfloor a_1\rfloor, \dots, \lfloor a_d\rfloor)$ with $\lfloor a_i\rfloor$ is the biggest integer less than or equal to a_i . In other words, the area on scaled grid will be represented by the bottom-left corner of the box. Note that scaling of the torus does not effect $\mathcal{R}_n$.
We recall the pair empirical measure on the scaled torus defined in (1.2.34):

$$L_{n,\epsilon} = \epsilon n^{-\frac{d-2}{d}}\sum_{i=1}^{\frac{1}{\epsilon}n^{\frac{d-2}{d}}}\delta_{\left(n^{-1/d}\mathcal{S}_{(i-1)\epsilon n^{2/d}},\,n^{-1/d}\mathcal{S}_{i\epsilon n^{2/d}}\right)}.$$

Let $I^{(2)}_\epsilon : \mathcal{M}^+_1(\Lambda_N \times \Lambda_N) \to [0,\infty]$ be the entropy function

$$I^{(2)}_\epsilon(\mu) = \begin{cases} h(\mu|\mu_1\otimes\pi_\epsilon) & \text{if } \mu_1 = \mu_2\\ \infty & \text{otherwise,}\end{cases} \tag{2.2.24}$$

where $\mathcal{M}_1^+$ denotes a probability measure space on Λ_N with weak topology, $h(\cdot|\cdot)$ denotes relative entropy between measures, μ_1 and μ_2 are the two marginals of μ and $\pi_\epsilon(x, dy) = p_\epsilon^\pi(y-x)dy$ is the Brownian transition kernel on Λ_N where $\mu_1 \otimes \pi_\epsilon := \int \mu_1(dx)\pi_\epsilon(x, dy)$. By (2.1.5) and (2.1.6), we get

$$p_t^\pi(x, y) = \sum_{z \in \mathbb{Z}^d} p_t(x, y + zN), \tag{2.2.25}$$

with $p_t^\pi(x) = p_t^\pi(0, x)$. Furthermore, for $\eta > 0$ let $\Phi_\eta : \mathcal{M}_1^+(\Lambda_N \times \Lambda_N) \to [0, \infty)$ be the function

$$\Phi_\eta(\mu) = \int_{\Lambda_N} dx \Big(1 - \exp\big[-2\eta\kappa \int_{\Lambda_N \times \Lambda_N} \varphi_\epsilon\big(y-x, z-x\big)\mu(dy, dz)\big]\Big), \tag{2.2.26}$$

with

$$\varphi_\epsilon(y, z) = \frac{\int_0^\epsilon ds p_{s/d}^\pi(-y) p_{(\epsilon-s)/d}^\pi(z)}{p_{\epsilon/d}^\pi(z-y)}. \tag{2.2.27}$$

Our main result in this section is the following proposition:

Proposition 2.2.4. *$\mathbb{E}_{n,\epsilon}\frac{1}{n}\mathcal{R}_n$ satisfies a LDP on $\mathbb{R}^+$ with speed $n^{\frac{d-2}{d}}$ and rate function*

$$J_{\epsilon/d}(b) = \inf\Big\{\frac{1}{\epsilon} I_{\epsilon/d}^{(2)}(\mu) : \mu \in \mathcal{M}_1^+(\Lambda_N \times \Lambda_N), \Phi_{1/\epsilon}(\mu) = b\Big\}.$$

Proof. Although we prove results on Λ_N, it is more natural to carry on the proof on the non-scaled torus, $\Lambda_{Nn^{1/d}}$, corresponding to the probability law $\mathbb{P}$.

Let $c_1, c_2, \ldots$ be constants that may depend on ϵ, N (which are fixed) but not on any of the other variables.

(1) We first approximate $\mathcal{R}_n$ by cutting small holes around the points $\mathcal{S}_{i\epsilon n^{2/d}}, 1 \le i \le \frac{1}{\epsilon} n^{\frac{d-2}{d}}$. Fix $K > 0$, for $1 < i < \frac{1}{\epsilon} n^{\frac{d-2}{d}}$ let,

$$\mathcal{W}_i^K = \mathcal{W}_i \setminus \big\{\mathcal{S}_j : |\mathcal{S}_j - \mathcal{S}_{(i-1)\epsilon n^{2/d}}| < K \text{ or } |\mathcal{S}_j - \mathcal{S}_{i\epsilon n^{2/d}}| < K, (i-1)\epsilon n^{2/d} < j < i\epsilon n^{2/d}\big\}. \tag{2.2.28}$$

Also, define

$$\frac{1}{n}\mathcal{R}_n^K = \frac{1}{n}\sharp\Big\{\bigcup_{i=1}^{\frac{1}{\epsilon}n^{\frac{d-2}{d}}} \mathcal{W}_i^K\Big\}. \tag{2.2.29}$$

Note that, this cutting procedure corresponds to removing balls of size $Kn^{-1/d}$ on the scaled torus. Therefore, we have cut at most $\frac{1}{\epsilon} n^{\frac{d-2}{d}}$ times the number of points in a ball of radius K, hence

$$\frac{1}{n}\Big|\mathcal{R}_n - \mathcal{R}_n^K\Big| \leq \frac{1}{n}c_1\frac{1}{\epsilon}n^{\frac{d-2}{d}}(2K)^d = \frac{2^d c_1 K^d}{\epsilon n^{2/d}}, \tag{2.2.30}$$

which tends to zero as $n \to \infty$ and therefore is negligible. This cutting procedure is done in order to make the intersection between $\mathcal{W}_i^K$ and $\mathcal{W}_{i+1}^K$ unlikely which will be important in the next step.

(2) Define $\sigma = \min\{n : \mathcal{S}_n = 0\} = \min\{n : \tilde{\mathcal{S}}_n = 0\}$. For $y, z \in \Lambda_N$, define

$$\begin{aligned} q_{n,\epsilon}(y,z) &= P\big(\sigma \leq \epsilon n^{\frac{2}{d}} | \tilde{\mathcal{S}}_0 = y, \tilde{\mathcal{S}}_{\epsilon n^{\frac{2}{d}}} = z\big) \\ &= \mathbb{P}\big(\sigma \leq \epsilon n^{\frac{2}{d}} | \mathcal{S}_0 = yn^{\frac{1}{d}}, \mathcal{S}_{\epsilon n^{\frac{2}{d}}} = zn^{\frac{1}{d}}\big). \end{aligned} \tag{2.2.31}$$

We define $P(\cdot) = P_0(\cdot)$, $\mathbb{P}(\cdot) = \mathbb{P}_0(\cdot), P_a(\cdot) = P(\cdot|\tilde{\mathcal{S}}_0 = a), \mathbb{P}_{an^{1/d}}(\cdot) = \mathbb{P}(\cdot|\mathcal{S}_0 = an^{1/d})$, and bridge measures$P_{a,b}(\cdot) = P(\cdot|\tilde{\mathcal{S}}_0 = a, \tilde{\mathcal{S}}_{\epsilon n^{2/d}} = b), \mathbb{P}_{an^{1/d},bn^{1/d}}(\cdot) = \mathbb{P}(\cdot|\mathcal{S}_0 = an^{1/d}, \mathcal{S}_{\epsilon n^{2/d}} = bn^{1/d})$.

We can express $\mathbb{E}_{n,\epsilon}\frac{1}{n}\mathcal{R}_n^K$ in terms of $q_{n,\epsilon}(y,z)$ and the empirical measure $L_{n,\epsilon}$ as follows:

$$\begin{aligned} \mathbb{E}_{n,\epsilon}\frac{1}{n}\mathcal{R}_n^K &= \frac{1}{n}\sum_{x\in\Lambda_{Nn^{\frac{1}{d}}}}\Big(1 - \mathbb{P}_{n,\epsilon}\big(x \notin \bigcup_{i=1}^{\frac{1}{\epsilon}n^{\frac{d-2}{d}}} \mathcal{W}_i^K\big)\Big) \\ &= \frac{1}{n}\sum_{x\in\Lambda_{Nn^{\frac{1}{d}}}}\Big(1 - \mathbb{P}_{n,\epsilon}\big(\bigcap_{i=1}^{\frac{1}{\epsilon}n^{\frac{d-2}{d}}} \{x \notin \mathcal{W}_i^K\}\big)\Big) \\ &= \frac{1}{n}\sum_{x\in\Lambda_{Nn^{\frac{1}{d}}}}\Big(1 - \prod_{i=1}^{\frac{1}{\epsilon}n^{\frac{d-2}{d}}} \mathbb{P}_{n,\epsilon}\big(x \notin \mathcal{W}_i^K\big)\Big) \\ &= \frac{1}{n}\sum_{x\in\Lambda_{Nn^{\frac{1}{d}}}}\Big(1 - \exp\big(\sum_{i=1}^{\frac{1}{\epsilon}n^{\frac{d-2}{d}}} \log\big[1 - \mathbb{P}_{n,\epsilon}(x \in \mathcal{W}_i^K)\big]\big)\Big) \\ &= \int_{\Lambda_N} dx\Bigg(1 - \exp\Big(\frac{1}{\epsilon}n^{\frac{d-2}{d}}\int_{\Lambda_N\times\Lambda_N} \\ &\qquad L_{n,\epsilon}(dy,dz)\log\big[1 - q_{n,\epsilon}^{Kn^{-1/d}}(y - \lfloor xn^{\frac{1}{d}}\rfloor n^{-\frac{1}{d}}, z - \lfloor xn^{\frac{1}{d}}\rfloor n^{-\frac{1}{d}})\big]\Big)\Bigg), \end{aligned} \tag{2.2.32}$$

where

- the last equality comes from scaling the torus and using the empirical measure,
- for $\rho > 0$, we define $q_{n,\epsilon}^{\rho}(y,z) = q_{n,\epsilon}(y,z)$ if $y, z \notin B_\rho$, the centred ball of radius

ρ, and zero otherwise.

(3) We want to expand the logarithm and do an approximation. For this we need the following facts about random walk on Λ_N. Recall that κ is the exit probability from the origin.

Proposition 2.2.5. *(a)* $\lim_{K\to\infty} \limsup_{n\to\infty} \sup_{a,b\notin B_{Kn^{-1/d}}} q_{n,\epsilon}(a,b) = 0$.

(b) $\lim_{n\to\infty} \sup_{a,b\notin B_\rho} |n^{\frac{d-2}{d}} q_{n,\epsilon}(a,b) - 2\kappa\varphi_\epsilon(a,b)| = 0$ *for all* $0 < \rho < N/4$.

Remark: Due to the parity problem of random walk we may assume that a, b and $\epsilon n^{2/d}$ have the same parity.

Proof. We will divide the proof of Proposition 2.2.5 into three parts. We first show that, after scaling $\Lambda_{Nn^{1/d}}$ to Λ_N, (2.1.5) converges to a function of the Brownian transition kernel on Λ_N defined in (2.2.25). Then, we prove Proposition 2.2.5 (a) and (b) separately.

I. The local central limit theorem

Lemma 2.2.6. *For* $a, b \in \Lambda_N$ *with O-term independent of* a *and* b *and for any* $\alpha < 1 + \frac{2}{d}$,

$$\mathfrak{p}^\pi_{\epsilon n^{2/d}}(an^{1/d}, bn^{1/d}) = \frac{2}{n} p^\pi_{\epsilon/d}(a,b) + o(n^{-\alpha}),$$

which implies that for any $\varepsilon > 0$ *and sufficiently large* n, *we have*

$$\frac{2-\varepsilon}{n} p^\pi_{\epsilon/d}(a,b) \le \mathfrak{p}^\pi_{\epsilon n^{2/d}}(an^{1/d}, bn^{1/d}) \le \frac{2+\varepsilon}{n} p^\pi_{\epsilon/d}(a,b).$$

Proof. We first show that $\mathfrak{p}^\pi_{\epsilon n^{2/d}}(an^{1/d}, bn^{1/d}) \le \frac{2}{n} p^\pi_{\epsilon/d}(a,b) + o(n^{-\alpha})$. By (2.1.7), for fixed m,

$$\begin{aligned}
&\mathfrak{p}^\pi_{\epsilon n^{2/d}}(an^{1/d}, bn^{1/d}) \\
&= \sum_{z\in\mathbb{Z}^d:||z||_\infty\le m} \mathfrak{p}_{\epsilon n^{2/d}}\big(an^{1/d}, (b+zN)n^{1/d}\big) + \sum_{z\in\mathbb{Z}^d:||z||_\infty> m} \mathfrak{p}_{\epsilon n^{2/d}}\big(an^{1/d}, (b+zN)n^{1/d}\big).
\end{aligned} \tag{2.2.33}$$

Now, by (2.1.8),

$$\begin{aligned}
&\sum_{z\in\mathbb{Z}^d:||z||_\infty\le m} \mathfrak{p}_{\epsilon n^{2/d}}\big(an^{1/d},(b+zN)n^{1/d}\big)\\
&\qquad=\sum_{z\in\mathbb{Z}^d:||z||_\infty\le m} 2\Big(\frac{d}{2\pi\epsilon n^{2/d}}\Big)^{d/2}\exp\Big[\frac{-d|(b+zN)n^{1/d}-an^{1/d}|^2}{2\epsilon n^{2/d}}\Big]\\
&\qquad\quad+(2m+1)^dO(n^{-1-\frac{2}{d}})\\
&\qquad=\sum_{z\in\mathbb{Z}^d:||z||_\infty\le m}\frac{2}{n}\Big(\frac{d}{2\pi\epsilon}\Big)^{d/2}\exp\Big[\frac{-d|(b+zN)-a|^2}{2\epsilon}\Big]+(2m+1)^dO\big(n^{-1-\frac{2}{d}}\big)\\
&\qquad\le\sum_{z\in\mathbb{Z}^d}\frac{2}{n}\Big(\frac{d}{2\pi\epsilon}\Big)^{d/2}\exp\Big[\frac{-d|(b+zN)-a|^2}{2\epsilon}\Big]+(3m)^dO\big(n^{-1-\frac{2}{d}}\big)\\
&\qquad=\frac{2}{n}p^{\pi}_{\epsilon/d}(b-a)+(3m)^dO\big(n^{-1-\frac{2}{d}}\big).
\end{aligned}\tag{2.2.34}$$

Now, for the second term of (2.2.33), by (2.1.11) and (2.1.12) for some constant $c_{d,\epsilon}$ depending on d and ϵ,

$$\begin{aligned}
\sum_{z\in\mathbb{Z}^d:||z||_\infty>m}\mathfrak{p}_{\epsilon n^{2/d}}\big(an^{1/d},(b+zN)n^{1/d}\big)&\le\frac{c_{d,\epsilon}}{n}\sum_{k=m}^{\infty}k^d\exp\Big[-\frac{k^2N^2}{\epsilon}\Big]\\
&\le\frac{\hat{c}_{d,\epsilon}}{n}m^d\exp\Big[-\frac{m^2N^2}{\epsilon}\Big].
\end{aligned}\tag{2.2.35}$$

Combining (2.2.33), (2.2.34) and (2.2.35) we get

$$\mathfrak{p}^{\pi}_{\epsilon n^{2/d}}(an^{1/d},bn^{1/d})\le\frac{2}{n}p^{\pi}_{\epsilon/d}(b-a)+(3m)^dO(n^{-1-\frac{2}{d}})+\frac{\hat{c}_{d,\epsilon}}{n}m^d\exp\Big[-\frac{m^2N^2}{\epsilon}\Big].\tag{2.2.36}$$

Now to minimise the error terms in (2.2.36), we choose $m=\sqrt{\frac{2\epsilon}{dN^2}\log n}$, this would give the required upper bound.

Next, we show that $\mathfrak{p}^{\pi}_{\epsilon n^{2/d}}(an^{1/d}, bn^{1/d}) \geq \frac{2-\varepsilon}{n} p^{\pi}_{\epsilon/d}(a,b) + o(n^{-\alpha})$. Write

$$\begin{aligned}
&\mathfrak{p}^{\pi}_{\epsilon n^{2/d}}(an^{1/d}, bn^{1/d}) \\
&\quad = \sum_{z \in \mathbb{Z}^d} \mathfrak{p}_{\epsilon n^{2/d}}\big(an^{1/d}, (b+zN)n^{1/d}\big) \\
&\quad \geq \sum_{z \in \mathbb{Z}^d : ||z||_\infty \leq m} \mathfrak{p}_{\epsilon n^{2/d}}\big(an^{1/d}, (b+zN)n^{1/d}\big) \\
&\quad = \sum_{z \in \mathbb{Z}^d : ||z||_\infty \leq m} \frac{2}{n}\Big(\frac{d}{2\pi\epsilon}\Big)^{d/2} \exp\left[\frac{-d|(b+zN)-a|^2}{2\epsilon}\right] + (2m)^d O(n^{-1-\frac{2}{d}}) \\
&\quad \geq \sum_{z \in \mathbb{Z}^d} \frac{2}{n}\Big(\frac{d}{2\pi\epsilon}\Big)^{d/2} \exp\left[\frac{-d|(b+zN)-a|^2}{2\epsilon}\right] + (2m)^d O(n^{-1-\frac{2}{d}}) \\
&\qquad - \sum_{z \in \mathbb{Z}^d : ||z||_\infty > m} \frac{2}{n}\Big(\frac{d}{2\pi\epsilon}\Big)^{d/2} \exp\left[\frac{-d|(b+zN)-a|^2}{2\epsilon}\right] \\
&\quad \geq \frac{2}{n} p^{\pi}_{\epsilon/d}(b-a) + \Big((2m)^d O(n^{-1-\frac{2}{d}}) - \frac{c'_{d,\epsilon}}{n} m^d \exp\Big[-\frac{m^2N^2}{\epsilon}\Big]\Big). \qquad (2.2.37)
\end{aligned}$$

Again, we choose $m = \sqrt{\frac{\varepsilon}{N^2}\log\frac{c_{d,\epsilon}}{2^d} + \frac{2}{d}\log n} = \sqrt{\frac{2}{d}\log c'_{d,\epsilon,N} n}$ and we get the required lower bound. $\square$

II. Proof of Proposition 2.2.5 (a)

Throughout the proof, ϵ, N are fixed.

(a) We begin by proving another lemma.

Let

- $P^m_{a,b}$ be the law of $\tilde{\mathcal{S}}_m$ under $P_{a,b}$ on $\Lambda_N \cap n^{-\frac{1}{d}}\mathbb{Z}^d$
- P^m_a be the law of $\tilde{\mathcal{S}}_m$ under P_a on $\Lambda_N \cap n^{-\frac{1}{d}}\mathbb{Z}^d$

Lemma 2.2.7. *For $m \in \big(0, \frac{1}{2}\epsilon n^{\frac{2}{d}}\big]$, there exists a constant c_2 such that for $a, b \in \Lambda_N$, $\frac{dP^m_{a,b}}{dP^m_a} \leq c_2$.*

Proof. Indeed, for $x \in \Lambda_N \cap n^{-\frac{1}{d}}\mathbb{Z}^d$,

$$\begin{aligned}
\frac{dP^m_{a,b}}{dP^m_a}(x) &= \frac{P_{a,b}(\tilde{\mathcal{S}}_m = x)}{P_a(\tilde{\mathcal{S}}_m = x)} \\
&= \frac{1}{P_a(\tilde{\mathcal{S}}_m = x)}\left[\frac{P_a(\tilde{\mathcal{S}}_m = x, \tilde{\mathcal{S}}_{\epsilon n^{2/d}} = b)}{P_a(\tilde{\mathcal{S}}_{\epsilon n^{2/d}} = b)}\right] \\
&= \frac{P_x(\tilde{\mathcal{S}}_{\epsilon n^{2/d} - m} = b)}{P_a(\tilde{\mathcal{S}}_{\epsilon n^{2/d}} = b)},
\end{aligned}$$

by the Markov property. Set $m = \tilde{m}\epsilon n^{2/d}$ with $0 < \tilde{m} < \frac{1}{2}$ and unscale Λ_N to $\Lambda_{Nn^{1/d}}$ we get

$$\frac{P_x(\tilde{\mathcal{S}}_{\epsilon n^{2/d}-m} = b)}{P_a(\tilde{\mathcal{S}}_{\epsilon n^{2/d}} = b)} = \frac{\mathbb{P}_{xn^{1/d}}(\mathcal{S}_{(1-\tilde{m})\epsilon n^{2/d}} = bn^{1/d})}{\mathbb{P}_{an^{1/d}}(\mathcal{S}_{\epsilon n^{2/d}} = bn^{1/d})}$$
$$= \frac{\mathfrak{p}^{\pi}_{(1-\tilde{m})\epsilon n^{2/d}}(xn^{1/d}, bn^{1/d})}{\mathfrak{p}^{\pi}_{\epsilon n^{2/d}}(an^{1/d}, bn^{1/d})}.$$

Now, using Lemma 2.1.2 and (2.1.8) we get

$$\begin{aligned}\frac{\mathfrak{p}^{\pi}_{(1-\tilde{m})\epsilon n^{2/d}}(xn^{1/d}, bn^{1/d})}{\mathfrak{p}^{\pi}_{\epsilon n^{2/d}}(an^{1/d}, bn^{1/d})} &\le 3^d \left[\frac{\mathfrak{p}_{(1-\tilde{m})\epsilon n^{2/d}}(xn^{1/d}, b^* n^{1/d})}{\mathfrak{p}_{\epsilon n^{2/d}}(an^{1/d}, bn^{1/d})}\right] + O(n^{-1}) \\ &= 3^d (1-\tilde{m})^{-d/2} \exp\left[\frac{d}{2\epsilon}(|b-a|^2 - |b^* - x|^2)\right] \\ &\quad + O(n^{-1-\frac{2}{d}}) + O(n^{-1}) \\ &\le 3^d \exp\left[\frac{d}{2\epsilon} N^2\right] + O(n^{-1}),\end{aligned}$$

since ϵ, d and N are fixed, $0 < \tilde{m} < \frac{1}{2}$ and $|b - a| < N$. Hence, we get the lemma. □

(b) We start the proof of Proposition 2.2.5 (a) by removing both the bridge restriction and the torus restriction. Let D_r be points on $\Lambda_N \cap n^{-1/d}\mathbb{Z}^d$ that are at most $\lfloor rn^{1/d} \rfloor$ steps away from the origin, and define ∂D_r be points that are exactly $\lfloor rn^{1/d} \rfloor$ steps away from the origin. For $a, b \in \Lambda_N$ and $0 < r < N/2$, let σ_r be the first entrance time into D_r. Also, recall that $P_a(\cdot) = P(\cdot | \tilde{\mathcal{S}}_0 = a)$ and $\sigma = \{\min n : \tilde{\mathcal{S}}_n = 0\}$. Next, we use Lemma 2.2.7, to deduce that, when n is large enough, for all $0 < Kn^{-\frac{1}{d}} < N/4$,

$$\begin{aligned}\sup_{a,b \notin D_{Kn^{-1/d}}} q_{n,\epsilon}(a,b) &\le \sup_{a,b \notin D_{Kn^{-1/d}}} P_{a,b}\Big(\sigma \le \frac{1}{2}\epsilon n^{\frac{2}{d}}\Big) + \sup_{a,b \notin D_{Kn^{-1/d}}} P_{b,a}\Big(\sigma \le \frac{1}{2}\epsilon n^{\frac{2}{d}}\Big) \\ &\le 2c_2 \sup_{a \notin D_{Kn^{-1/d}}} P_a\Big(\sigma \le \frac{1}{2}\epsilon n^{\frac{2}{d}}\Big). \end{aligned} \tag{2.2.38}$$

Now, let $\hat{\sigma}$ be the first entrance time into $D^c_{N/2} = \Lambda_N \setminus D_{N/2}$. We may write that

$$P_a\Big(\sigma \le \frac{1}{2}\epsilon n^{\frac{2}{d}}\Big) = P_a\Big(\sigma \le \frac{1}{2}\epsilon n^{\frac{2}{d}}, \sigma < \hat{\sigma}\Big) + P_a\Big(\sigma \le \frac{1}{2}\epsilon n^{\frac{2}{d}}, \sigma > \hat{\sigma}\Big). \tag{2.2.39}$$

To estimate the second term of the RHS of (2.2.39), we note that, by choice on n, on its way from $\partial D_{N/2}$ to the origin, the walk must first cross $\partial D_{N/4}$ and then

$\partial D_{Kn^{-1/d}}$. Hence, by the strong Markov property, for any $a \notin D_{Kn^{-1/d}}$

$$P_a\Big(\sigma \le \frac{1}{2}\epsilon n^{\frac{2}{d}}, \sigma > \hat{\sigma}\Big) \le c_3 \sup_{x\in\partial D_{Kn^{-1/d}}} P_x\Big(\sigma \le \frac{1}{2}\epsilon n^{\frac{2}{d}}\Big). \tag{2.2.40}$$

where $c_3 = \sup_n \sup_{x\in\partial D_{N/2}} P_x\big(\sigma_{\frac{N}{4}n^{1/d}} < \frac{1}{2}\epsilon n^{\frac{2}{d}}\big)$. Evidently $c_3 < 1$ and we can deduce that

$$\sup_{a\notin D_{Kn^{-1/d}}} P_a\Big(\sigma \le \frac{1}{2}\epsilon n^{\frac{2}{d}}\Big) \le \frac{1}{1-c_3} \sup_{a\notin D_{Kn^{-1/d}}} P_a\Big(\sigma \le \frac{1}{2}\epsilon n^{\frac{2}{d}}, \sigma < \hat{\sigma}\Big). \tag{2.2.41}$$

As long as the walk does not hit $D^c_{N/2}$, it behaves like random walk on $n^{-1/d}\mathbb{Z}^d$. Define $\tau = \min\{n : S_n = 0\} = \min\{n : n^{-1/d}S_n = 0\}$. Hence,

$$P_a\Big(\sigma \le \frac{1}{2}\epsilon n^{\frac{2}{d}}, \sigma < \hat{\sigma}\Big) \le P_a\Big(\tau \le \frac{1}{2}\epsilon n^{\frac{2}{d}}\Big). \tag{2.2.42}$$

Combining (2.2.38), (2.2.41) and (2.2.42) we get, for all $0 < Kn^{-1/d} < N/4$,

$$\sup_{a\notin D_{Kn^{-1/d}}} q_{n,\epsilon}(a,b) \le \frac{c_2}{1-c_3} \sup_{a\notin D_{Kn^{-1/d}}} P_a\Big(\tau \le \frac{1}{2}\epsilon n^{\frac{2}{d}}\Big). \tag{2.2.43}$$

(c) Now, we unscale the random walk on Λ_N to $\Lambda_{Nn^{1/d}}$. Similarly, we define $\tilde{D}_r$ to be points on $\Lambda_{Nn^{1/d}} \cap \mathbb{Z}^d$ that are at most $\lfloor rn^{1/d} \rfloor$ steps away from the origin. Using Lemma 2.1.1 (b) we get

$$\begin{aligned}
\sup_{a\notin D_{Kn^{-1/d}}} P_a\Big(\tau \le \frac{1}{2}\epsilon n^{\frac{2}{d}}\Big) &= \sup_{u\notin \tilde{D}_{Kn^{-1/d}}} \mathbb{P}\Big(\tau \le \frac{1}{2}\epsilon n^{\frac{2}{d}} | S_0 = u\Big)\\
&\le \mathbb{P}\Big(\tau \le \frac{1}{2}\epsilon n^{\frac{2}{d}} \big| |S_0| = K\Big)\\
&\le \mathbb{P}(\tau < \infty \big| |S_0| = K)\\
&\le \mathbb{P}(\mathcal{S}_n = 0 \text{ for some } n \big| |S_0| = K)\\
&\le \frac{c_4 + o(1)}{K^{d-2}}.
\end{aligned}$$

Hence,

$$\sup_{a\notin D_{Kn^{-1/d}}} q_{n,\epsilon}(a,b) \le \frac{c_2}{1-c_3}\Big(\frac{c_4+o(1)}{K^{d-2}}\Big). \tag{2.2.44}$$

Now, note that $D_{Kn^{-1/d}} \subset B_{Kn^{-1/d}}$, and therefore $\sup_{a\notin B_{Kn^{-1/d}}} q_{n,\epsilon}(a,b) \le \sup_{a\notin D_{Kn^{-1/d}}} q_{n,\epsilon}(a,b)$. Hence, we can conclude that,

$$\sup_{a\notin B_{Kn^{-1/d}}} q_{n,\epsilon}(a,b) \le \frac{c_2}{1-c_3}\Big(\frac{c_4+o(1)}{K^{d-2}}\Big).$$

By taking the limit $n \to \infty$ followed by $K \to \infty$, we get the proof of Proposition 2.2.5 (a).

III. Proof of Proposition 2.2.5(b)

Again, we set ϵ, N fixed.

a) From Lemma 2.2.6 for any $\varepsilon > 0$ and $\alpha < 1 + \frac{2}{d}$, with o-term independent of a and b,

$$\mathfrak{p}^{\pi}_{\epsilon n^{2/d}}(an^{1/d}, bn^{1/d}) = \frac{2}{n} p^{\pi}_{\epsilon/d}(a,b) + o(n^{-\alpha}). \tag{2.2.45}$$

Now, by the similar arguments as in Lemma 2.2.6, and setting $k = sn^{2/d}$ for $s < \epsilon$, we can also show that

$$\mathfrak{p}^{\pi}_{(\epsilon-s)n^{2/d}}(bn^{\frac{1}{d}}) = \frac{2}{n} p^{\pi}_{(\epsilon-s)/d}(b) + o\big(n^{-\alpha}\big), \tag{2.2.46}$$
$$\mathfrak{p}^{\pi}_{sn^{2/d}}(-an^{\frac{1}{d}}) = \frac{2}{n} p^{\pi}_{s/d}(-a) + o\big(n^{-\alpha}\big). \tag{2.2.47}$$

b) Let $0 < \delta < \epsilon/2$. For $a, b \in \Lambda_N \cap (n^{-1/d}\mathbb{Z}^d)$, define

$$q_{n,\epsilon}(a, b, \delta n^{\frac{2}{d}}) = P_{a,b}\Big(\delta n^{\frac{2}{d}} < \sigma < (\epsilon - \delta) n^{\frac{2}{d}}\Big). \tag{2.2.48}$$

Note that

$$\begin{aligned}
&\sup_{a,b \notin B_\rho} \Big| n^{\frac{d-2}{d}} q_{n,\epsilon}(a,b) - n^{\frac{d-2}{d}} q_{n,\epsilon}(a,b,\delta n^{\frac{2}{d}}) \Big| \\
&\qquad \le n^{\frac{d-2}{d}} \sup_{a,b \notin B_\rho} \Big(P_{a,b}\big(\sigma \le \delta n^{\frac{2}{d}}\big) + P_{a,b}\big((\epsilon - \delta) n^{\frac{2}{d}} \le \sigma < \epsilon n^{\frac{2}{d}}\big)\Big). \\
&\qquad \le 2n^{\frac{d-2}{d}} \sup_{a,b \notin B_\rho} P_{a,b}\big(\sigma < \delta n^{\frac{2}{d}}\big) \\
&\qquad \le \frac{2c_2}{1-c_3} n^{\frac{d-2}{d}} \sup_{a \notin B_\rho} P_a\big(\tau < \delta n^{\frac{2}{d}}\big),
\end{aligned} \tag{2.2.49}$$

where the last equality comes from (2.2.43). Now, using the Markov property and Lemma 2.1.1 (a) under the condition $\delta < |\rho|^2$,

$$\begin{aligned}
\sup_{a \notin B_\rho} P_a\big(\tau < \delta n^{\frac{2}{d}}\big) &\le \sum_{k=1}^{\delta n^{\frac{2}{d}}} \mathbb{P}\big(S_k = 0 | S_0 = \rho n^{1/d}\big) \\
&\le \sum_{k=1}^{\delta n^{\frac{2}{d}}} \bigg| O\bigg(k^{-d/2} \exp\Big[-\frac{|\rho n^{1/d}|^2}{2k}\Big]\bigg)\bigg|.
\end{aligned} \tag{2.2.50}$$

Note that $O\big(k^{-d/2}\exp\big[-|\rho n^{1/d}|^2/2k\big]\big)$ is maximised when $k=|\rho|^2 n^{2/d}/d$. Hence,

$$\begin{aligned}\sum_{k=1}^{\delta n^{\frac{2}{d}}}\left|O\left(k^{-d/2}\exp\left[-\frac{|\rho n^{1/d}|^2}{2k}\right]\right)\right| &\le \delta n^{2/d} O\left(\frac{1}{n}\Big(\frac{|\rho|^2}{d}\Big)^{-d/2}\exp\Big[-\frac{d}{2}\Big]\right)\\ &\le \delta n^{-1+\frac{2}{d}} O\left(\Big(\frac{|\rho|^2}{d}\Big)^{-d/2}\exp\Big[-\frac{d}{2}\Big]\right). \qquad (2.2.51)\end{aligned}$$

Therefore, by (2.2.49), (2.2.50) and (2.2.51),

$$\sup_{a,b\notin B_\rho}\big|n^{\frac{d-2}{d}}q_{n,\epsilon}(a,b)-n^{\frac{d-2}{d}}q_{n,\epsilon}(a,b,\delta n^{\frac{2}{d}})\big| \le \frac{2c_2}{1-c_3}\delta\Big(\frac{|\rho|^2}{d}\Big)^{d/2}\exp\Big[-\frac{d}{2}\Big]O(1).$$

Hence,

$$\lim_{\delta\downarrow 0}\lim_{n\to\infty}\sup_{a,b\notin B_\rho}\big|n^{\frac{d-2}{d}}q_{n,\epsilon}(a,b)-n^{\frac{d-2}{d}}q_{n,\epsilon}(a,b,\delta n^{2/d})\big|=0.$$

So, Proposition 2.2.5 (b) can be proved by replacing $q_{n,\epsilon}(a,b)$ by $q_{n,\epsilon}\big(a,b,\delta n^{2/d}\big)$.
c) We need instead to show that

$$\lim_{\delta\downarrow 0}\lim_{n\to\infty}\big|n^{\frac{d-2}{d}}q_{n,\epsilon}(a,b,\delta n^{2/d})-2\kappa\varphi_\epsilon(a,b)\big|=0. \qquad (2.2.52)$$

Note that

$$\begin{aligned} n^{\frac{d-2}{d}}q_{n,\epsilon}\big(a,b,\delta n^{2/d}\big) &= n^{\frac{d-2}{d}}\sum_{k=\delta n^{\frac{2}{d}}}^{(\epsilon-\delta)n^{\frac{2}{d}}}\mathbb{P}_{an^{\frac{1}{d}},bn^{\frac{1}{d}}}(\sigma=k)\\ &= n^{\frac{d-2}{d}}\sum_{k=\delta n^{\frac{2}{d}}}^{(\epsilon-\delta)n^{\frac{2}{d}}}\frac{\mathbb{P}_{an^{\frac{1}{d}}}\big(\mathcal{S}_k=0,\mathcal{S}_1,\dots,\mathcal{S}_{k-1}\neq 0\big)\cdot\mathbb{P}_0\big(\mathcal{S}_{\epsilon n^{\frac{2}{d}}-k}=bn^{\frac{1}{d}}\big)}{\mathbb{P}_{an^{\frac{1}{d}}}\big(\mathcal{S}_{\epsilon n^{\frac{2}{d}}}=bn^{\frac{1}{d}}\big)}. \end{aligned} \qquad (2.2.53)$$

By using Lemma 2.2.8 below, the proof of Proposition 2.2.5 (b) follows.

Lemma 2.2.8.

$$\begin{aligned}\lim_{n\to\infty}\Bigg|n^{\frac{d-2}{d}}\sum_{k=\delta n^{\frac{2}{d}}}^{(\epsilon-\delta)n^{\frac{2}{d}}}&\frac{\mathbb{P}_{an^{\frac{1}{d}}}\big(\mathcal{S}_k=0,\mathcal{S}_1,\dots,\mathcal{S}_{k-1}\neq 0\big)\mathbb{P}_0\big(\mathcal{S}_{\epsilon n^{\frac{2}{d}}-k}=bn^{\frac{1}{d}}\big)}{\mathbb{P}_{an^{\frac{1}{d}}}\big(\mathcal{S}_{\epsilon n^{\frac{2}{d}}}=bn^{\frac{1}{d}}\big)}\\ &-2\kappa\int_\delta^{\epsilon-\delta}ds\,\frac{p^\pi_{s/d}(-a)p^\pi_{(\epsilon-s)/d}(b)}{p^\pi_{\epsilon/d}(b-a)}\Bigg|=0.\end{aligned}$$

We will prove upper and lower bound separately. We denote **(A)** by

$$\mathbf{(A)} := n^{\frac{d-2}{d}} \sum_{k=\delta n^{\frac{2}{d}}}^{(\epsilon-\delta)n^{\frac{2}{d}}} \frac{\mathbb{P}_{an^{\frac{1}{d}}}\big(\mathcal{S}_k = 0, \mathcal{S}_1, \ldots, \mathcal{S}_{k-1} \neq 0\big) \cdot \mathbb{P}_0\big(\mathcal{S}_{\epsilon n^{\frac{2}{d}}-k} = bn^{\frac{1}{d}}\big)}{\mathbb{P}_{an^{\frac{1}{d}}}\big(\mathcal{S}_{\epsilon n^{\frac{2}{d}}} = bn^{\frac{1}{d}}\big)}.$$

d)

Proof of Lemma 2.2.8: The upper bound. We shall show that

$$\limsup_{n\to\infty} \mathbf{(A)} \leq 2\kappa \int_{\delta}^{\epsilon-\delta} ds \frac{p^{\pi}_{s/d}(-a)p^{\pi}_{(\epsilon-s)/d}(b)}{p^{\pi}_{\epsilon/d}(b-a)}. \tag{2.2.54}$$

For n large enough, let $l(< k) \in \mathbb{N}$ fixed. Note that by reversing the random walk and then using the Markov property:

$$\begin{aligned}
\mathbf{(A)} &\leq n^{\frac{d-2}{d}} \sum_{k=\delta n^{\frac{2}{d}}}^{(\epsilon-\delta)n^{\frac{2}{d}}} \frac{\mathbb{P}_0\big(\mathcal{S}_k = -an^{\frac{1}{d}}, \mathcal{S}_1, \ldots, \mathcal{S}_l \neq 0\big) \cdot \mathbb{P}_0\big(\mathcal{S}_{\epsilon n^{\frac{2}{d}}-k} = bn^{\frac{1}{d}}\big)}{\mathbb{P}_{an^{\frac{1}{d}}}\big(\mathcal{S}_{\epsilon n^{\frac{2}{d}}} = bn^{\frac{1}{d}}\big)} \\
&\leq n^{\frac{d-2}{d}} \sum_{k=\delta n^{\frac{2}{d}}}^{(\epsilon-\delta)n^{\frac{2}{d}}} \frac{\mathbb{E}_0\big[\mathbf{1}\{\mathcal{S}_1, \ldots, \mathcal{S}_l \neq 0\} \cdot \hat{\mathbb{P}}_{\mathcal{S}_l}\big(\hat{\mathcal{S}}_{k-l} = -an^{-\frac{1}{d}}\big)\big] \cdot \mathbb{P}_0\big(\mathcal{S}_{\epsilon n^{\frac{2}{d}}-k} = bn^{\frac{1}{d}}\big)}{\mathbb{P}_{an^{\frac{1}{d}}}\big(\mathcal{S}_{\epsilon n^{\frac{2}{d}}} = bn^{\frac{1}{d}}\big)},
\end{aligned} \tag{2.2.55}$$

where under $\hat{\mathbb{P}}_x$ the random walk $(\hat{\mathcal{S}}_m)_{1\leq m\leq k-l}$ is independent of $(\mathcal{S}_j)$ and started in x. We can see that

$$\mathbb{P}_x\big(\hat{\mathcal{S}}_{k-l} = -an^{\frac{1}{d}}\big) \leq \mathbb{P}_0\big(\hat{\mathcal{S}}_{k-l} = -an^{\frac{1}{d}} - x\big) = \mathfrak{p}^{\pi}_{k-l}(-an^{1/d} - x).$$

Now, by (2.1.7) and (2.1.8), for $\varepsilon', \varepsilon'' > 0$, we get hold of $n_0 \in \mathbb{N}$ such that for all $n > n_0$,

$$\begin{aligned}
\mathfrak{p}^{\pi}_{k-l}(-an^{1/d} - x) &= \sum_{z\in\mathbb{Z}^d} \left[2\bigg(\frac{d}{2\pi(k-l)}\bigg)^{d/2} \exp\Big[\frac{-d}{2(k-l)}|(a+zN)n^{\frac{1}{d}} + x|^2\Big] + \tilde{\tilde{A}}_n(a)\right] \\
&\leq \sum_{z\in\mathbb{Z}^d} \left[2\Big(\frac{d}{2\pi k}\Big)^{d/2}(1+\varepsilon')\exp\Big[\frac{-d}{2k}|-(a+zN)n^{\frac{1}{d}}|^2(1-\varepsilon'')\Big] + \tilde{\tilde{A}}_n(a)\right] \\
&\leq \sum_{z\in\mathbb{Z}^d} \mathfrak{p}^{\pi}_{k/(1-\varepsilon'')}\big(-(a+zN)n^{\frac{1}{d}}\big) \times (1+\varepsilon') \\
&\leq (1+\varepsilon')\mathfrak{p}^{\pi}_{k/(1-\varepsilon'')}(-an^{\frac{1}{d}}),
\end{aligned}$$

where, for $k \geq \delta n^{2/d}$,

- $(1+\varepsilon') \leq \left(\frac{k}{k-l}\right)^{d/2} = (1+\frac{l}{k-l})^{d/2} \overset{n\to\infty}{\longrightarrow} 1$,
- $(1-\varepsilon'') \geq \frac{|an^{1/d}+x|^2}{|an^{1/d}|^2}\frac{k}{k-l} \overset{n\to\infty}{\longrightarrow} 1$ uniformly on $x \in \{a : |a| < l\}$.

Note that, we abused the notation $\mathfrak{p}^{\pi}_{k/(1-\varepsilon'')}(\cdot)$ since the time may not be an integer. However, we prefer to use this terminology since it is clear to see what will happen when we pass the limit $\varepsilon \downarrow 0$. The formula in (2.1.2) can also be used to give an approximation of $\mathfrak{p}^{\pi}_t(\cdot)$ for non-integer t.

Also, note that the error term $\tilde{\tilde{A}}_n(a)$ from the local central limit theorem in (2.1.2) does not change the order when we insert ε' and ε''.
From (2.2.55), taking supremum limit on both sides and using Lemma 2.2.6 we get

$$
\begin{aligned}
&\limsup_{n\to\infty} \mathbf{(A)} \\
&\leq \limsup_{n\to\infty} n^{\frac{d-2}{d}} \sum_{k=\delta n^{\frac{2}{d}}}^{(\epsilon-\delta)n^{\frac{2}{d}}} \left(\frac{(1+\varepsilon')\mathfrak{p}^{\pi}_{k/(1-\varepsilon'')}(-an^{\frac{1}{d}})\mathfrak{p}^{\pi}_{\epsilon n^{2/d}-k}(bn^{1/d})}{\mathfrak{p}^{\pi}_{\epsilon n^{2/d}}(bn^{1/d}-an^{1/d})} \right) \times \mathbb{P}_0\big(\mathcal{S}_1,\dots,\mathcal{S}_l \neq 0\big) \\
&\leq \limsup_{n\to\infty} \left(n^{\frac{d-2}{d}} \int_{\delta n^{\frac{2}{d}}}^{(\epsilon-\delta)n^{\frac{2}{d}}} dk \frac{(1+\varepsilon')\mathfrak{p}^{\pi}_{k/(1-\varepsilon'')}(-an^{\frac{1}{d}})\mathfrak{p}^{\pi}_{\epsilon n^{2/d}-k}(bn^{1/d})}{\mathfrak{p}^{\pi}_{\epsilon n^{2/d}}(bn^{1/d}-an^{1/d})} \mathbb{P}_0\big(\mathcal{S}_1,\dots,\mathcal{S}_l \neq 0\big) \right) \\
&= \limsup_{n\to\infty} \left(n \int_{\delta}^{\epsilon-\delta} ds \frac{(1+\varepsilon')\mathfrak{p}^{\pi}_{sn^{2/d}/(1-\varepsilon'')}(-an^{1/d})\mathfrak{p}^{\pi}_{(\epsilon-s)n^{2/d}}(bn^{1/d})}{\mathfrak{p}^{\pi}_{\epsilon n^{2/d}}(bn^{1/d}-an^{1/d})} \mathbb{P}_0\big(\mathcal{S}_1,\dots,\mathcal{S}_l \neq 0\big) \right) \\
&\leq \limsup_{n\to\infty} \left(2 \int_{\delta}^{\epsilon-\delta} ds \frac{p^{\pi}_{s/d}(-a)p^{\pi}_{(\epsilon-s)/d}(b)}{p^{\pi}_{\epsilon/d}(b-a)} \mathbb{P}_0\big(\mathcal{S}_1,\dots,\mathcal{S}_l \neq 0\big) + o\big(n^{-\alpha}\big) \right) \\
&= 2 \int_{\delta}^{\epsilon-\delta} ds \frac{p^{\pi}_{s/d}(-a)p^{\pi}_{(\epsilon-s)/d}(b)}{p^{\pi}_{\epsilon/d}(b-a)} \mathbb{P}_0\big(\mathcal{S}_1,\dots,\mathcal{S}_l \neq 0\big).
\end{aligned}
$$

Finally, letting $l \to \infty$ we then get,

$$
\lim_{l\to\infty} \limsup_{n\to\infty} \mathbf{(A)} \leq 2\kappa \int_{\delta}^{\epsilon-\delta} ds \frac{p^{\pi}_{s/d}(-a)p^{\pi}_{(\epsilon-s)/d}(b)}{p^{\pi}_{\epsilon/d}(b-a)}.
$$

e) *Proof of Lemma 2.2.8: The lower bound*
We shall show that

$$
\liminf_{n\to\infty} \mathbf{(A)} \geq 2\kappa \int_{\delta}^{\epsilon-\delta} ds \frac{p^{\pi}_{s/d}(-a)p^{\pi}_{(\epsilon-s)/d}(b)}{p^{\pi}_{\epsilon/d}(b-a)}. \tag{2.2.56}
$$

Set $\frac{4}{d^2} < \gamma < \frac{2}{d}$ and $\theta = \delta' n^{\gamma}$ for $\delta' < \delta$. Also, define

$$\textbf{(B)} := n^{\frac{d-2}{d}} \sum_{k=\delta n^{\frac{2}{d}}}^{(\epsilon-\delta)n^{\frac{2}{d}}} \frac{\mathbb{P}_0\big(\mathcal{S}_k = -an^{\frac{1}{d}}, \mathcal{S}_1, \ldots, \mathcal{S}_\theta \neq 0\big) \cdot \mathbb{P}_0\big(\mathcal{S}_{\epsilon n^{\frac{2}{d}}-k} = bn^{\frac{1}{d}}\big)}{\mathbb{P}_{an^{\frac{1}{d}}}\big(\mathcal{S}_{\epsilon n^{\frac{2}{d}}} = bn^{\frac{1}{d}}\big)},$$

$$\textbf{(C)} := n^{\frac{d-2}{d}} \sum_{k=\delta n^{\frac{2}{d}}}^{(\epsilon-\delta)n^{\frac{2}{d}}} \frac{\mathbb{P}_0\big(\mathcal{S}_k = -an^{\frac{1}{d}}, \mathcal{S}_j = 0 \text{ for some } \theta < j < k\big) \cdot \mathbb{P}_0\big(\mathcal{S}_{\epsilon n^{\frac{2}{d}}-k} = bn^{\frac{1}{d}}\big)}{\mathbb{P}_{an^{\frac{1}{d}}}\big(\mathcal{S}_{\epsilon n^{\frac{2}{d}}} = bn^{\frac{1}{d}}\big)}.$$

Then,

$$\textbf{(A)} \geq \textbf{(B)} - \textbf{(C)}.$$

Our aim is to show that,

$$\liminf_{n\to\infty} \textbf{(B)} \geq 2\kappa \int_{\delta}^{\epsilon-\delta} ds \frac{p^{\pi}_{s/d}(-a)p^{\pi}_{(\epsilon-s)/d}(b)}{p^{\pi}_{\epsilon/d}(b-a)}, \tag{2.2.57}$$

$$\limsup_{n\to\infty} \textbf{(C)} = 0. \tag{2.2.58}$$

*Proof of **(2.2.57)***

Let $0 < \beta < \frac{1}{2}$. Now,

$$\begin{aligned}
\textbf{(B)} &= n^{\frac{d-2}{d}} \sum_{k=\delta n^{\frac{2}{d}}}^{(\epsilon-\delta)n^{\frac{2}{d}}} \frac{\mathbb{E}_0\Big(\mathbf{1}\{\mathcal{S}_1, \ldots, \mathcal{S}_\theta \neq 0\} \cdot \hat{\mathbb{P}}_{\mathcal{S}_\theta}\big(\hat{\mathcal{S}}_{k-\theta} = -an^{\frac{1}{d}}\big)\Big) \cdot \mathbb{P}_0\big(\mathcal{S}_{\epsilon n^{\frac{2}{d}}-k} = bn^{\frac{1}{d}}\big)}{\mathbb{P}_{an^{\frac{1}{d}}}\big(\mathcal{S}_{\epsilon n^{\frac{2}{d}}} = bn^{\frac{1}{d}}\big)} \\
&\geq \frac{n^{\frac{d-2}{d}}}{\mathbb{P}_{an^{\frac{1}{d}}}\big(\mathcal{S}_{\epsilon n^{\frac{2}{d}}} = bn^{\frac{1}{d}}\big)} \sum_{k=\delta n^{\frac{2}{d}}}^{(\epsilon-\delta)n^{\frac{2}{d}}} \mathbb{E}_0\Big[\Big(\mathbf{1}\{\mathcal{S}_1, \ldots, \mathcal{S}_\theta \neq 0\} \times \mathbf{1}\{|\mathcal{S}_\theta| < \theta^{\frac{1}{2}+\beta}\} \\
&\quad \times \inf_{|c| \leq \theta^{\frac{1}{2}+\epsilon}} \hat{\mathbb{P}}_c\big(\hat{\mathcal{S}}_{k-\theta} = -an^{\frac{1}{d}}\big)\Big) \times \mathbb{P}_0\big(\mathcal{S}_{\epsilon n^{\frac{2}{d}}-k} = bn^{\frac{1}{d}}\big)\Big].
\end{aligned}$$

Now, by using the same technique as in the upper bound case, we can see that for $\varepsilon', \varepsilon'' > 0$ and $k < \epsilon n^{2/d}$, there exists n_0 such that for all $n > n_0$,

$$\inf_{|c| \leq \theta^{\frac{1}{2}+\beta}} \hat{\mathbb{P}}_c\big(\hat{\mathcal{S}}_{k-\theta} = -an^{\frac{1}{d}}\big) \geq \mathfrak{p}^{\pi}_{k/(1+\varepsilon'')}(-an^{\frac{1}{d}}) \times (1 - \varepsilon').$$

Hence,

$$\begin{aligned}
\mathbf{(B)} &\geq \frac{n^{\frac{d-2}{d}}}{\mathfrak{p}^{\pi}_{\epsilon n^{2/d}}(bn^{1/d}-an^{1/d})} \sum_{k=\delta n^{\frac{2}{d}}}^{(\epsilon-\delta)n^{\frac{2}{d}}} \mathbb{E}_0\Big[\Big(\mathbf{1}\{\mathcal{S}_1,\ldots,\mathcal{S}_\theta\neq 0\} - \mathbf{1}\{|\mathcal{S}_\theta| > \theta^{\frac{1}{2}+\beta}\}\Big) \\
&\qquad\qquad \times \mathfrak{p}^{\pi}_{k/(1+\varepsilon'')}(-an^{\frac{1}{d}}) \cdot (1-\varepsilon') \cdot \mathfrak{p}^{\pi}_{\epsilon n^{2/d}-k}(bn^{1/d})\Big] \\
&= \frac{n^{\frac{d-2}{d}}\Big(\mathbb{P}\big(\mathcal{S}_1,\ldots,\mathcal{S}_\theta\neq 0\big) - \mathbb{P}\big(|\mathcal{S}_\theta| > \theta^{\frac{1}{2}+\beta}\big)\Big)}{\mathfrak{p}^{\pi}_{\epsilon n^{2/d}}(bn^{1/d}-an^{1/d})} \\
&\qquad \times \sum_{k=\delta n^{\frac{2}{d}}}^{(\epsilon-\delta)n^{\frac{2}{d}}} \Big(\mathfrak{p}^{\pi}_{k/(1+\varepsilon'')}(-an^{\frac{1}{d}}) \cdot (1-\varepsilon') \cdot \mathfrak{p}^{\pi}_{\epsilon n^{2/d}-k}(bn^{1/d})\Big). \qquad (2.2.59)
\end{aligned}$$

Next, we need to estimate the probability $\mathbb{P}\big(|\mathcal{S}_\theta| > \theta^{\frac{1}{2}+\beta}\big)$. This can be done by the moderate deviation principle (e.g. see [deA92]). Let $(S_n)_{n\in\mathbb{N}}$ be a d-dimensional random walk. Assume $\sqrt{n} \ll a_n \ll n$ in the sense of (1.2.23). Then, for $x > 0$,

$$\lim_{n\to\infty} \frac{n}{a_n^2} \log \mathbb{P}(|S_n| \geq x a_n) = -\frac{x^2}{2}. \qquad (2.2.60)$$

By using (2.2.60) and the Markov property, we can deduce that for large n and $\beta < \frac{1}{2}$,

$$\mathbb{P}\big(|\mathcal{S}_\theta| > \theta^{\frac{1}{2}+\beta}\big) \leq \mathbb{P}\big(|S_\theta| > \theta^{\frac{1}{2}+\beta}\big) \approx \exp(\frac{-\theta^{2\beta}}{2}),$$

in the sense of (2.2.3). Moreover,

$$\lim_{n\to\infty} n^{\frac{d-2}{d}} e^{-\frac{n^{2\beta}}{2}} = 0.$$

Combine (2.2.59) with Lemma 2.2.6 and the same argument as in the upper bound case, we get

$$\begin{aligned}
&\liminf_{n\to\infty} \mathbf{(B)} \\
&\geq \liminf_{n\to\infty} \mathbb{P}\big(\mathcal{S}_1,\ldots,\mathcal{S}_\theta \neq 0\big)\Bigg[n^{\frac{d-2}{d}} \sum_{k=\delta n^{\frac{2}{d}}}^{(\epsilon-\delta)n^{\frac{2}{d}}} \frac{\mathfrak{p}^{\pi}_{k/(1+\epsilon'')}(-an^{\frac{1}{d}}) \cdot (1-\epsilon') \cdot \mathfrak{p}^{\pi}_{\epsilon n^{2/d}-k}(bn^{1/d})}{\mathfrak{p}^{\pi}_{\epsilon n^{2/d}}(bn^{1/d}-an^{1/d})}\Bigg] \\
&\geq \liminf_{n\to\infty} \Bigg[2\kappa \int_{\delta}^{\epsilon-\delta} ds \frac{p^{\pi}_{s/d}(-a) p^{\pi}_{(\epsilon-s)/d}(b)}{p^{\pi}_{\epsilon/d}(b-a)} + o\big(n^{-\alpha}\big)\Bigg] \\
&\geq 2\kappa \int_{\delta}^{\epsilon-\delta} ds \frac{p^{\pi}_{s/d}(-a) p^{\pi}_{(\epsilon-s)/d}(b)}{p^{\pi}_{\epsilon/d}(b-a)}.
\end{aligned}$$

*Proof of **(2.2.58)***

By rewriting **(C)** and using Markov property,

$$\begin{aligned}
\mathbf{(C)} &\le n^{\frac{d-2}{d}} \sum_{k=\delta n^{\frac{2}{d}}}^{(\epsilon-\delta)n^{\frac{2}{d}}} \frac{\sum_{j=\theta}^{k} \mathbb{P}_0\big(S_j = 0, S_k = -an^{\frac{1}{d}}\big) \cdot \mathbb{P}_0\big(S_{\epsilon n^{\frac{2}{d}}-k} = bn^{\frac{1}{d}}\big)}{\mathbb{P}_{an^{\frac{1}{d}}}\big(S_{\epsilon n^{\frac{2}{d}}} = bn^{\frac{1}{d}}\big)} \\
&= n^{\frac{d-2}{d}} \sum_{k=\delta n^{\frac{2}{d}}}^{(\epsilon-\delta)n^{\frac{2}{d}}} \frac{\Big(\sum_{j=\theta}^{k} \mathbb{P}_0\big(S_j = 0\big) \cdot \mathbb{P}_0\big(S_{k-j} = -an^{\frac{1}{d}}\big)\Big) \cdot \mathbb{P}_0\big(S_{\epsilon n^{\frac{2}{d}}-k} = bn^{\frac{1}{d}}\big)}{\mathbb{P}_{an^{\frac{1}{d}}}\big(S_{\epsilon n^{\frac{2}{d}}} = bn^{\frac{1}{d}}\big)} \\
&= n^{\frac{d-2}{d}} \int_{\delta n^{\frac{2}{d}}}^{(\epsilon-\delta)n^{\frac{2}{d}}} dk \frac{\Big(\int_{\theta}^{k} dj \mathbb{P}_0\big(S_j = 0\big) \cdot \mathbb{P}_0\big(S_{k-j} = -an^{\frac{1}{d}}\big)\Big) \cdot \mathbb{P}_0\big(S_{\epsilon n^{\frac{2}{d}}-k} = bn^{\frac{1}{d}}\big)}{\mathbb{P}_0\big(S_{\epsilon n^{\frac{2}{d}}} = (b-a)n^{\frac{1}{d}}\big)},
\end{aligned}$$

Setting $k = sn^{2/d}$ and using the fact that $\mathbb{P}(S_j = 0) \le Cj^{-\frac{d}{2}}$ for some $j \in \mathbb{N}$ for a constant C, we get

$$\begin{aligned}
\mathbf{(C)} &\le n \int_{\delta}^{(\epsilon-\delta)} ds \frac{\mathfrak{p}^{\pi}_{(\epsilon-s)n^{2/d}}(bn^{1/d})}{\mathfrak{p}^{\pi}_{\epsilon n^{2/d}}\big((b-a)n^{1/d}\big)} \bigg(\int_{\theta}^{sn^{\frac{2}{d}}} dj \quad \mathbb{P}_0\big(S_j = 0\big) \cdot \mathbb{P}_0\big(S_{sn^{2/d}-j} = -an^{\frac{1}{d}}\big) \bigg) \\
&\le n \int_{\delta}^{(\epsilon-\delta)} ds \frac{\mathfrak{p}^{\pi}_{(\epsilon-s)n^{2/d}}(bn^{1/d})}{\mathfrak{p}^{\pi}_{\epsilon n^{2/d}}\big((b-a)n^{1/d}\big)} \int_{\theta}^{sn^{\frac{2}{d}}} dj \quad Cj^{-\frac{d}{2}} \cdot \mathbb{P}_0\big(S_{sn^{2/d}-j} = -an^{\frac{1}{d}}\big).
\end{aligned}$$

Again, set $j = xn^{2/d}$,we get

$$\begin{aligned}
&\mathbf{(C)} \\
&\le n^{\frac{2}{d}} \int_{\delta}^{(\epsilon-\delta)} ds \frac{\mathfrak{p}^{\pi}_{(\epsilon-s)n^{2/d}}(bn^{1/d})}{\mathfrak{p}^{\pi}_{\epsilon n^{2/d}}\big((b-a)n^{1/d}\big)} \int_{\delta' n^{\gamma-\frac{2}{d}}}^{s} dx \quad Cx^{-\frac{d}{2}} \cdot \mathfrak{p}^{\pi}_{(s-x)n^{2/d}}(-an^{1/d}) \\
&\le Cn^{\frac{2}{d}} \int_{\delta}^{(\epsilon-\delta)} ds \frac{\mathfrak{p}^{\pi}_{(\epsilon-s)n^{2/d}}(bn^{1/d})}{\mathfrak{p}^{\pi}_{\epsilon n^{2/d}}\big((b-a)n^{1/d}\big)} \int_{\delta' n^{\gamma-\frac{2}{d}}}^{s} dx \quad \mathfrak{p}^{\pi}_{(s-x)n^{2/d}}(-an^{1/d}) \times [\delta' n^{\gamma-\frac{2}{d}}]^{-\frac{d}{2}} \\
&= Cn^{1+\frac{2}{d}-\frac{\gamma d}{2}} \int_{\delta}^{(\epsilon-\delta)} ds \frac{\mathfrak{p}^{\pi}_{(\epsilon-s)n^{2/d}}(bn^{1/d})}{\mathfrak{p}^{\pi}_{\epsilon n^{2/d}}\big((b-a)n^{1/d}\big)} \int_{\delta' n^{\gamma-\frac{2}{d}}}^{s} dx \mathfrak{p}^{\pi}_{(s-x)n^{2/d}}(-an^{1/d}). \qquad (2.2.61)
\end{aligned}$$

For the last integral in (2.2.61) , by using Lemma 2.2.6,

$$\begin{aligned}
\int_{\delta' n^{\gamma-\frac{2}{d}}}^{s} dx \mathfrak{p}^{\pi}_{(s-x)n^{2/d}}(-an^{1/d}) &= \frac{2}{n} \int_{\delta' n^{\gamma-\frac{2}{d}}}^{s} dx p^{\pi}_{s-x}(-a) + o(n^{-\alpha}) \\
&\le \frac{2}{n} s + o(n^{-\alpha}) \le \frac{2}{n} \epsilon + o(n^{-\alpha}), \qquad (2.2.62)
\end{aligned}$$

where ϵ is fixed and note that $p^{\pi}_{s-x}(\cdot)$ is a transition probability. Moreover, the first integral in (2.2.61) is bounded (This can be checked by direct substitution using (2.1.2), (2.1.5) and Lemma 2.1.2). Combining this with (2.2.61) and (2.2.62), for another constant $\hat{C}$, we get

$$\textbf{(C)} \leq \hat{C} n^{\frac{2}{d}-\frac{\gamma d}{2}}.$$

Recall that $\frac{4}{d^2} < \gamma < \frac{2}{d}$, we then claim that $\limsup_{n\to\infty} \textbf{(C)} = 0$. Hence, from (2.2.57) and (2.2.58), we can deduce (2.2.56). Combining the upper bound and the lower bound, we can deduce Lemma 2.2.8. □

Hence, we have completed the proof of Proposition 2.2.5 (b). □

(4) We start from where we left off at the end of step 2, see page 56. We want to modify (2.2.32) using Proposition 2.2.5. For $y, z \in \Lambda_N$, recall that $q^{\rho}_{n,\epsilon}(y,z) = q_{n,\epsilon}(y,z)$ if $y, z \notin B_\rho$, the ball of radius ρ, and zero otherwise. From Proposition 2.2.5(a), it follows that there exists $\delta_K > 0$, satisfying $\lim_{K\to\infty} \delta_K = 0$, such that the inequalities

$$-(1+\delta_K) q^{Kn^{-\frac{1}{d}}}_{n,\epsilon}(y,z) \leq \log\Big(1 - q^{Kn^{-\frac{1}{d}}}_{n,\epsilon}(y,z)\Big) \leq -q^{Kn^{-\frac{1}{d}}}_{n,\epsilon}(y,z), \qquad (2.2.63)$$

hold for all y, z and for $n \in \mathbb{N}$ large enough. Now we introduce the function

$$\Phi_{n,\eta,\rho} : \mathcal{M}^+_1(\Lambda_N \times \Lambda_N) \longrightarrow [0, \infty),$$

defined by

$$\Phi_{n,\eta,\rho}(\mu) = \int_{\Lambda_N} dx \Big(1 - \exp\Big[-\eta n \int_{\Lambda_N\times\Lambda_N} q^{\rho}_{n,\epsilon}(y-x, z-x)\mu(dy,dz)\Big]\Big). \qquad (2.2.64)$$

The main advantage of introducing the function $\Phi_{n,\eta,\rho}$ is we can get good upper and lower bounds. By using (2.2.32) and (2.2.64), we get:

$$\Phi_{n^{\frac{d-2}{d}},1/\epsilon,Kn^{-\frac{1}{d}}}(L_{n,\epsilon}) \leq \mathbb{E}_{n,\epsilon} \tfrac{1}{n}\mathcal{R}^K_n \leq \Phi_{n^{\frac{d-2}{d}},(1+\delta_K)/\epsilon,Kn^{-\frac{1}{d}}}(L_{n,\epsilon}) \qquad (2.2.65)$$

Moreover, the function also provide nice upper bounds for its approximation.

Lemma 2.2.9. *There exists constants c_4, c_5 such that:*

(a) $|\Phi_{n^{\frac{d-2}{d}},\eta,\rho}(\mu) - \Phi_{n^{\frac{d-2}{d}},\eta,\rho'}(\mu)| \leq c_4 \eta |\rho^2 + \rho'^2|$ *for all η, μ.*

(b) $|\Phi_{n^{\frac{d-2}{d}},\eta,\rho}(\mu) - \Phi_{n^{\frac{d-2}{d}},\eta',\rho}(\mu)| \leq c_5 |\eta - \eta'|$ *for all ρ, μ and $n \geq n_0(\rho)$.*

Proof. Define $\varphi^{\rho}_{\epsilon}(y,z) = \varphi_\epsilon(y,z)$ if $y, z \notin B_\rho$ and zero otherwise. Note that for $x, y > 0$, $|e^{-x} - e^{-y}| \leq |x - y|$. Here, $o_{\rho,\rho'}(1)$ means an error tending to zero as $n \to \infty$ depending on ρ, ρ' but not on other variables.

(a) Let $B_\rho(y)$ be the ball of radius ρ centred at y. By using the triangle inequality and

Proposition 2.2.5 (b),

$$\begin{aligned}
&\left|\Phi_{n^{\frac{d-2}{d}},\eta,\rho}(\mu) - \Phi_{n^{\frac{d-2}{d}},\eta,\rho'}(\mu)\right| \\
&\leq \eta \int_{\Lambda_N} dx \int_{\Lambda_N \times \Lambda_N} \mu(dy,dz) \left| n^{\frac{d-2}{d}} q^{\rho}_{n,\epsilon}(y-x,z-x) - n^{\frac{d-2}{d}} q^{\rho'}_{n,\epsilon}(y-x,z-x)\right| \\
&= \eta \int_{\Lambda_N} dx \int_{\Lambda_N \times \Lambda_N} \mu(dy,dz) \Big[\left| 2\kappa \varphi^{\rho}_{\epsilon}(y-x,z-x) - 2\kappa \varphi^{\rho'}_{\epsilon}(y-x,z-x)\right| + o_{\rho,\rho'}(1) \Big]
\end{aligned}$$

$$\begin{aligned}
&\leq 2\eta\kappa \int_{\Lambda_N} dx \int_{\Lambda_N \times \Lambda_N} \mu(dy,dz) \left| \varphi^{\rho}_{\epsilon}(y-x,z-x) - \varphi_{\epsilon}(y-x,z-x)\right| \\
&\quad + 2\eta\kappa \int_{\Lambda_N} dx \int_{\Lambda_N \times \Lambda_N} \mu(dy,dz) \left| \varphi^{\rho'}_{\epsilon}(y-x,z-x) - \varphi_{\epsilon}(y-x,z-x)\right| + |\Lambda_N| o_{\rho,\rho'}(1).
\end{aligned} \tag{2.2.66}$$

Therefore, we can calculate the two integrals in (2.2.66) separately. Note that

$$\begin{aligned}
&\int_{\Lambda_N} dx \int_{\Lambda_N \times \Lambda_N} \mu(dy,dz) |\varphi^{\rho}_{\epsilon}(y-x,z-x) - \varphi_{\epsilon}(y-x,z-x)| \\
&\qquad \leq \int_{\Lambda_N} dx \int_{(B_\rho(x)\times\Lambda_N)\cup(\Lambda_N \times B_\rho(x))} \mu(dy,dz) \varphi_{\epsilon}(y-x,z-x) \\
&\qquad \leq \int_{\Lambda_N \times \Lambda_N} \mu(dy,dz) \int_{B_\rho(y)} dx \int_0^{\epsilon} ds \frac{p^{\pi}_{s/d}(x-y) p^{\pi}_{(\epsilon-s)/d}(z-x)}{p^{\pi}_{\epsilon/d}(z-y)} \\
&\qquad\quad + \int_{\Lambda_N \times \Lambda_N} \mu(dy,dz) \int_{B_\rho(z)} dx \int_0^{\epsilon} ds \frac{p^{\pi}_{s/d}(x-y) p^{\pi}_{(\epsilon-s)/d}(z-x)}{p^{\pi}_{\epsilon/d}(z-y)},
\end{aligned} \tag{2.2.67}$$

at which both integral can be estimated in the same way. Now, we consider the first term of the integral above by splitting the range of the integral as follows:

$$\begin{aligned}
&\int_{B_\rho(y)} dx \int_0^{\epsilon} ds \frac{p^{\pi}_{s/d}(x-y) p^{\pi}_{(\epsilon-s)/d}(z-x)}{p^{\pi}_{\epsilon/d}(z-y)} \\
&\qquad = \int_{B_\rho(y)} dx \int_0^{\epsilon/2} ds \frac{p^{\pi}_{s/d}(x-y) p^{\pi}_{(\epsilon-s)/d}(z-x)}{p^{\pi}_{\epsilon/d}(z-y)} \\
&\qquad\quad + \int_{B_\rho(y)} dx \int_{\epsilon/2}^{\epsilon} ds \frac{p^{\pi}_{s/d}(x-y) p^{\pi}_{(\epsilon-s)/d}(z-x)}{p^{\pi}_{\epsilon/d}(z-y)}.
\end{aligned} \tag{2.2.68}$$

Now, for the first term of (2.2.68), note that as s comes close to zero, the terms $p^{\pi}_{\epsilon/d}(\cdot)$ and $p^{\pi}_{(\epsilon-s)/d}(\cdot)$ can be approximated by a constant C. Also, by modifying Lemma 2.1.2

to get the upper bound of the integral, we can write this term as:

$$\int_{B_\rho(y)} dx \int_0^{\epsilon/2} ds \frac{p^\pi_{s/d}(x-y)p^\pi_{(\epsilon-s)/d}(z-x)}{p^\pi_{\epsilon/d}(z-y)} \le C \int_0^{\epsilon/2} ds \int_{B_\rho(0)} dx p^\pi_{s/d}(x)$$
$$\le 3^d C \int_0^{\epsilon/2} ds \int_{B_\rho(0)} dx p_{s/d}(x)$$
$$\le 3^d C^* \int_0^{\epsilon/2} ds \int_{B_\rho(0)} dx \frac{1}{s^{d/2}} e^{-\frac{d|x|^2}{2s}}. \tag{2.2.69}$$

From (2.2.69), we separate the integral into two cases:

- $|x|^2 \le s/d$.
  Then, we have

$$\int_0^{\epsilon/2} ds \int_{\{x \in B_\rho(0):|x|^2<s/d\}} dx \frac{1}{s^{d/2}} e^{-\frac{d|x|^2}{2s}} \le \int_0^{\epsilon/2} ds \int_{B_{\min(\rho,\sqrt{s/d})}(0)} dx \frac{1}{s^{d/2}}$$
$$\le \int_0^{d\rho^2} \frac{1}{d^{d/2}} ds + \rho^d \int_{d\rho^2}^{\epsilon/2} ds \frac{1}{s^{d/2}}$$
$$\le d^{1-\frac{d}{2}} \rho^2 - \rho^d [s^{1-d/2}]^{\epsilon/2}_{d\rho^2} = O(\rho^2).$$

- $|x|^2 > s/d$.
  Then, we have

$$\int_0^{\epsilon/2} ds \int_{\{x \in B_\rho(0):|x|^2>s/d\}} dx \frac{1}{s^{d/2}} e^{-\frac{d|x|^2}{2s}} \le \int_{B_\rho(0)} dx \int_0^{d|x|^2} \frac{1}{s^{d/2}} e^{-\frac{d|x|^2}{2s}} ds$$

By changing the variables from s to $\tilde{s} := \frac{s}{d|x|^2}$, we get

$$\int_{B_\rho(0)} dx \int_0^{d|x|^2} \frac{1}{s^{d/2}} e^{-\frac{d|x|^2}{2s}} ds \le d^{1-\frac{d}{2}} \int_{B_\rho(0)} dx \Big[|x|^{2-d} \int_0^1 \frac{1}{\tilde{s}^{d/2}} e^{-\frac{1}{2\tilde{s}}} d\tilde{s} \Big]$$
$$\le d^{1-\frac{d}{2}} \int_{B_\rho(0)} (d|x|)^{2-d} dx$$
$$\le d^{1-\frac{d}{2}} \int_0^\rho r^{d-1} r^{2-d} dr$$
$$\le d^{1-\frac{d}{2}} \int_0^\rho r dr = O(\rho^2).$$

Therefore, we can conclude that

$$\int_{B_\rho(y)} dx \int_0^{\epsilon/2} ds \frac{p^\pi_{s/d}(x-y)p^\pi_{(\epsilon-s)/d}(z-x)}{p^\pi_{\epsilon/d}(z-y)} = O(\rho^2).$$

Secondly, we look at the second term of (2.2.68). This time, the terms $p^\pi_{\epsilon/d}(\cdot)$ and $p^\pi_{s/d}(\cdot)$ can be approximated by constants. We can use the same arguments as above to assert that

$$\int_{B_\rho(y)} dx \int_{\epsilon/2}^{\epsilon} ds \frac{p^\pi_{s/d}(x-y)p^\pi_{(\epsilon-s)/d}(z-x)}{p^\pi_{\epsilon/d}(z-y)} = O(\rho^2).$$

Therefore we can estimate the integral in (2.2.66) as $c_4\eta\rho^2$.
(b) We use Proposition 2.2.5(b).

$$\begin{aligned}
&|\Phi_{n^{\frac{d-2}{d}},\eta,\rho}(\mu) - \Phi_{n^{\frac{d-2}{d}},\eta',\rho}(\mu)| \\
&\qquad \le |\eta-\eta'| \int_{\Lambda_N} dx \int_{\Lambda_N\times\Lambda_N} \mu(dy,dz) n^{\frac{d-2}{d}} q^\rho_{n,\epsilon}(y-x,z-x) \\
&\qquad = |\eta-\eta'| \int_{\Lambda_N} dx \int_{\Lambda_N\times\Lambda_N} \mu(dy,dz) \big[2\kappa\varphi^\rho_\epsilon(y-x,z-x) + o_\rho(1)\big] \\
&\qquad \le |\eta-\eta'|[2\kappa\epsilon + |\Lambda_N| o_\rho(1)],
\end{aligned}$$

where in the last inequality, we drop the superscript ρ to be able to perform the $x-$integration and use the fact that $\int_{\Lambda_N} dx\varphi_\epsilon(y-x,z-x) = \epsilon$. □

(5) We start to collect all the results together. By (2.2.30), (2.2.65) and Lemma 2.2.9 we get

$$\begin{aligned}
\mathbb{E}_{n,\epsilon}\frac{1}{n}\mathcal{R}_n &\le \mathbb{E}_{n,\epsilon}\frac{1}{n}\mathcal{R}^K_n + \frac{1}{n}c_1\big(\tfrac{1}{\epsilon}n^{\frac{d-2}{d}}K^d\big) \\
&\le \Phi_{n^{\frac{d-2}{d}},(1+\delta_K)/\epsilon,Kn^{-\frac{1}{d}}}(L_{n,\epsilon}) + \frac{c_1K^d}{\epsilon n^{\frac{2}{d}}} \\
&\le \Phi_{n^{\frac{d-2}{d}},1/\epsilon,\rho}(L_{n,\epsilon}) + \frac{c_1K^d}{\epsilon n^{\frac{2}{d}}} + c_4\frac{(Kn^{-1/d})^2+\rho^2}{\epsilon} + c_5\frac{\delta_K}{\epsilon}. \qquad (2.2.70)
\end{aligned}$$

and also a similar lower bound.

(6) Next, we approximate $\Phi_{n^{\frac{d-2}{d}},1/\epsilon,\rho}(L_{n,\epsilon})$ by another function $\Phi_{\infty,1/\epsilon,\rho}(L_{n,\epsilon})$ defined by

$$\Phi_{\infty,\eta,\rho}(\mu) = \int_{\Lambda_N} dx\Big(1-\exp\Big[-2\eta\kappa\int_{\Lambda_N\times\Lambda_N}\varphi^\rho_\epsilon(y-x,z-x)\mu(dy,dz)\Big]\Big). \qquad (2.2.71)$$

In order to continue from (2.2.70), we need the following Lemma:

Lemma 2.2.10. *There exists constants* $c_6, c_7, c_8 > 0$ *such that:*

(a) $|\Phi_{\infty,\eta,\rho}(\mu) - \Phi_{n^{\frac{d-2}{d}},\eta,\rho}(\mu)| \leq c_6 \eta \delta_{\rho,n}$ *for all* μ *with* $\lim_{n\to\infty} \delta_{\rho,n} = 0$ *for any* $\rho > 0$.

(b) $|\Phi_{\infty,\eta,\rho}(\mu) - \Phi_{\infty,\eta,0}(\mu)| \leq c_7 \eta \rho^2$ *for all* η, μ.

(c) $|\Phi_{\infty,1/\epsilon,0}(\mu) - \Phi_{\infty,1/\epsilon,0}(\mu')| \leq c_8 ||\mu - \mu'||_{tv}$ *where* $||\cdot||_{tv}$ *denotes the total variation norm.*

Proof. (a) We again use Proposition 2.2.5 (b).

$$\begin{aligned}
&\left|\Phi_{\infty,\eta,\rho}(\mu) - \Phi_{n^{\frac{d-2}{d}},\eta,\rho}(\mu)\right| \\
&\leq \eta \int_{\Lambda_N} dx \int_{\Lambda_N \times \Lambda_N} \mu(dy, dz) \left| n^{\frac{d-2}{d}} q^{\rho}_{n,\epsilon}(y - x, z - x) - 2\kappa \varphi^{\rho}_{\epsilon}(y - x, z - x)\right| \\
&= \eta \int_{\Lambda_N} dx \int_{\Lambda_N \times \Lambda_N} \mu(dy, dz) o_\rho(1) = \eta |\Lambda_N| o_\rho(1).
\end{aligned}$$

(b) By Proposition 2.2.5 (b),

$$\begin{aligned}
&\left|\Phi_{\infty,\eta,\rho}(\mu) - \Phi_{\infty,\eta,0}(\mu)\right| \\
&\qquad \leq \eta\kappa \int_{\Lambda_N} dx \int_{\Lambda_N \times \Lambda_N} \mu(dy, dz) \left|\varphi^{\rho}_{\epsilon}(y - x, z - x) - \varphi_{\epsilon}(y - x, z - x)\right| \\
&\qquad \leq \eta\kappa \int_{\Lambda_N} dx \int_{(\Lambda_N \times B_\rho(x)) \cup (B_\rho(x) \times \Lambda_N)} \mu(dy, dz) \varphi_{\epsilon}(y - x, z - x).
\end{aligned}$$

Now, we can use the same arguments as in Lemma 2.2.9(a) to claim the lemma.

(c) For $|\mu| = \mu^+ + \mu^-$ the variation of μ,

$$\begin{aligned}
&\left|\Phi_{\infty,1/\epsilon,0}(\mu) - \Phi_{\infty,1/\epsilon,0}(\mu')\right| \\
&\qquad\qquad \leq \frac{2\kappa}{\epsilon} \int_{\Lambda_N} dx \int_{\Lambda_N \times \Lambda_N} |\mu - \mu'|(dy, dz) \varphi_{\epsilon}(y - x, z - x) \\
&\qquad\qquad = 2\kappa \int_{\Lambda_N \times \Lambda_N} |\mu - \mu'|(dy, dz) = 2\kappa ||\mu - \mu'||_{tv}.
\end{aligned}$$

□

(7) Now, we do the final collection of results. By (2.2.70), the similar lower bound and

Lemma 2.2.10$(a),(b)$ with $\eta = 1/\epsilon$, we now have that for any K and ρ,

$$||\tfrac{1}{n}\mathbb{E}_{n,\epsilon}\mathcal{R}_n - \Phi_{\infty,1/\epsilon,0}(L_{n,\epsilon})||_\infty \leq \frac{c_1 K^d}{\epsilon n^{2/d}} + \frac{c_4(Kn^{-1/d})^2+\rho^2}{\epsilon} + \frac{c_5\delta_K}{\epsilon} + \frac{c_6\delta_{\rho,n}}{\epsilon} + \frac{c_7\rho^2}{\epsilon}. \quad (2.2.72)$$

Letting $n \to \infty$ followed by $K \to \infty$ and $\rho \downarrow 0$, we thus arrive at

$$\lim_{n\to\infty} ||\tfrac{1}{n}\mathbb{E}_{n,\epsilon}\mathcal{R}_n - \Phi_{\infty,1/\epsilon,0}(L_{n,\epsilon})||_\infty = 0 \text{ for all } \epsilon > 0. \quad (2.2.73)$$

(8) In order to complete the proof of Proposition 2.2.4, we need an LDP for the empirical pair measure of the skeleton walk. Other than the Wiener sausage case, the Donsker-Varadhan result does not apply here, as the Gaussian kernel of the skeleton walk is n-dependent. However, the following result shows that this dependence decays quick enough for an LDP to hold.

Proposition 2.2.11. *Let $(\tilde{\mathcal{S}}_n)_{n>0}$ be simple random walk on the torus $\Lambda_{Nn^{1/d}} \cap \mathbb{Z}^d$. Then, the empirical pair measure*

$$\hat{L}_{n,\epsilon} = \epsilon n^{-\frac{d-2}{d}} \sum_{i=1}^{\frac{1}{\epsilon}n^{\frac{d-2}{d}}} \delta_{(n^{-1/d}\tilde{\mathcal{S}}_{(i-1)\epsilon n^{2/d}},\, n^{-1/d}\tilde{\mathcal{S}}_{i\epsilon n^{2/d}})}$$

satisfies a LDP with speed $n^{\frac{d-2}{d}}$ and rate function $\frac{1}{\epsilon}I^{(2)}_{\epsilon/d}$ where $I^{(2)}_\epsilon$ is defined in (2.2.24).

Proof. Set $m := \frac{1}{\epsilon}n^{\frac{d-2}{d}}$ and let $s_0 = 0$. Let A be a measurable subset of $\mathcal{M}_1^+(\Lambda_N \times \Lambda_N)$, then

$$\begin{aligned}
&\mathbb{P}(\hat{L}_{n,\epsilon} \in A)\\
&= \sum_{s_1,\ldots,s_m \in \Lambda_N \cap n^{-1/d}\mathbb{Z}^d} \mathbf{1}_{\left\{\frac{1}{m}\sum_{j=1}^m \delta_{(s_{j-1},s_j)} \in A\right\}} \prod_{j=1}^m \mathbb{P}\big(\tilde{\mathcal{S}}_{j\epsilon n^{2/d}} = s_j n^{1/d} | \tilde{\mathcal{S}}_{(j-1)\epsilon n^{2/d}} = s_{j-1}n^{1/d}\big)\\
&= n^m \int_{\Lambda_N} \cdots \int_{\Lambda_N} ds_1 \ldots ds_m\\
&\qquad \mathbf{1}_{\left\{\frac{1}{m}\sum_{j=1}^m \delta_{(\lfloor s_{j-1}n^{1/d}\rfloor n^{-1/d},\, \lfloor s_j n^{1/d}\rfloor n^{-1/d})} \in A\right\}} \prod_{j=1}^m \mathbb{P}(\tilde{\mathcal{S}}_{\epsilon n^{2/d}} = \lfloor s_j n^{1/d}\rfloor - \lfloor s_{j-1}n^{1/d}\rfloor)\\
&= n^m \int_{\Lambda_N} \cdots \int_{\Lambda_N} ds_1 \ldots ds_m\\
&\qquad \mathbf{1}_{\left\{\frac{1}{m}\sum_{j=1}^m \delta_{(\lfloor s_{j-1}n^{1/d}\rfloor n^{-1/d},\, \lfloor s_j n^{1/d}\rfloor n^{-1/d})} \in A\right\}} \prod_{j=1}^m \mathfrak{p}^\pi_{\epsilon n^{2/d}}(\lfloor s_j n^{1/d}\rfloor - \lfloor s_{j-1}n^{1/d}\rfloor).
\end{aligned} \quad (2.2.74)$$

Note that from Lemma 2.2.6, for any $\varepsilon > 0$ and sufficiently large n,

$$\frac{2-\varepsilon}{n} p^{\pi}_{\epsilon/d}(s_j - s_{j-1}) \leq \mathfrak{p}^{\pi}_{\epsilon n^{2/d}}(s_j n^{1/d} - s_{j-1} n^{1/d}) \leq \frac{2+\varepsilon}{n} p^{\pi}_{\epsilon/d}(s_j - s_{j-1}),$$

provided that $s_j, s_{j-1} \in \Lambda_N \cap n^{-1/d}\mathbb{Z}^d$, and $\epsilon n^{2/d}$, $s_j n^{1/d}$, $s_{j-1} n^{1/d}$ have the same parity. However, if we use this to transform $\mathfrak{p}^{\pi}_{\epsilon n^{2/d}}(\lfloor s_j n^{1/d} \rfloor - \lfloor s_{j-1} n^{1/d} \rfloor)$ into $p^{\pi}_{\epsilon/d}(s_j - s_{j-1})$ under the integral in (2.2.74), this will cost a factor of a half. This is because we need to consider only points with the same parity. For any $\lfloor s_j \rfloor$, the set of points s_{j-1} such that $\lfloor s_{j-1} \rfloor$ and $\lfloor s_j \rfloor$ have the same parity is a "checker board" of cubes with sidelength $n^{-1/d}$ with total volume $\frac{1}{2}|\Lambda_N|$. Therefore, under the integral in (2.2.74), we only use half of the actual values. Combining with (2.2.74), we get

$$\left(1 - \tfrac{\varepsilon}{2}\right)^m \mathbb{P}(\tilde{L}_{n,\epsilon} \in A) \leq \mathbb{P}(\hat{L}_{n,\epsilon} \in A) \leq \left(1 + \tfrac{\varepsilon}{2}\right)^m \mathbb{P}(\tilde{L}_{n,\epsilon} \in A),$$

where

$$\mathbb{P}\{\tilde{L}_{n,\epsilon} \in A\} = \int_{\Lambda_N} \cdots \int_{\Lambda_N} ds_1 \dots ds_m \mathbf{1}_{\left\{\frac{1}{m}\sum_{j=1}^m \delta_{(s_{j-1},s_j)} \in A\right\}} \prod_{j=1}^m p^{\pi}_{\epsilon/d}(s_j - s_{j-1})$$

is the probability that the empirical pair measure $\tilde{L}_{n,\epsilon}$ of an m-step random walk with Gaussian transition kernel $p^{\pi}_{\epsilon/d}$ is in A. Therefore, Donsker-Varadhan theory can be applied (see [DV76] as well as [BBH01] pp. 377) and we can conclude that $\tilde{L}_{n,\epsilon}$ satisfies the LDP on $\mathcal{M}_1^+(\Lambda_N \times \Lambda_N)$ with speed $n^{\frac{d-2}{d}}$ and rate function $\frac{1}{\epsilon} I^{(2)}_{\epsilon/d}$.

□

(9) Finally, we can derive the desired large deviation principle for fixed ϵ as follows. Firstly, the function $\Phi_{\infty,1/\epsilon,0}$, defined in (2.2.71), is continuous in the total variation topology by Lemma 2.2.10(c). Also, note that $\Phi_{\infty,1/\epsilon,0}$ is exactly equals to $\Phi_{1/\epsilon}$, the function defined in (2.2.26). Now, by (2.2.73), we can combine the result from Proposition 2.2.11 along with the contraction principle to claim that $\frac{1}{n}\mathbb{E}_{n,\epsilon}\mathcal{R}_n$ satisfies the large deviation principle with the required speed and the required rate function in Proposition 2.2.4. □

2.2.3 The limit $\epsilon \downarrow 0$ for the LDP

In this section, we collect the results from [BBH01] to deduce the rate function when $\epsilon \downarrow 0$. Proofs of the results are omitted since we can directly use the proof of that in Section 2.4 [BBH01] where κ can be taken as an arbitrary number.

(a) We denote by $I : \mathcal{M}_1^+(\Lambda_N) \to [0, \infty]$ the standard large deviation rate function for the empirical distribution of the random walk:

$$I(\nu) = \begin{cases} \frac{1}{2}\int_{\Lambda_N} |\nabla\phi|^2(x)dx, & \text{if } \frac{d\mu}{dx} = \phi^2 \text{ with } \phi \in H^1(\Lambda_N) \\ \infty, & \text{otherwise.} \end{cases} \tag{2.2.75}$$

We further denote by $I_{\epsilon/d} : \mathcal{M}_1^+(\Lambda_N) \to [0,\infty]$ the following projection of $I_{\epsilon/d}^{(2)}$, defined in (2.2.24), onto $\mathcal{M}_1^+(\Lambda_N)$:

$$I_{\epsilon/d}(\nu) = \inf\{I_{\epsilon/d}^{(2)}(\mu) : \mu_1 = \nu\}. \tag{2.2.76}$$

Below is the result from Lemma 5 from [BBH01].

Lemma 2.2.12. *Let $(\pi_t)_{t\geq 0}$ denote the semigroup of Brownian motion. Then, for all $\nu \in \Lambda_N$, we have $t \mapsto I_t(\nu)/t$ is non-increasing, with $\lim_{t\downarrow 0} I_t(\nu)/t = I(\nu)$.*

(b) We need an approximation of the function $\Phi_{1/\epsilon} : \mathcal{M}_1^+(\Lambda_N \times \Lambda_N) \to [0,\infty)$, defined in (2.2.26), by the simpler functions $\Psi_{1/\epsilon} : \mathcal{M}_1^+(\Lambda_N) \to [0,\infty)$ defined by

$$\Psi_{1/\epsilon}(\nu) = \int_{\Lambda_N} dx\Big(1 - \exp\Big[-\frac{2\kappa}{\epsilon}\int_0^\epsilon ds \int_{\Lambda_N} p_s^\pi(x-y)\nu(dy)\Big]\Big). \tag{2.2.77}$$

Below is the result from Lemma 6 from [BBH01].

Lemma 2.2.13. *For $\mu \in \mathcal{M}_1^+(\Lambda_N \times \Lambda_N)$, and for any $K > 0$,*

$$\lim_{\epsilon\downarrow 0} \sup_{\mu: \frac{d}{\epsilon} I_{\epsilon/d}^{(2)}(\mu) \leq K} \big|\Phi_{1/\epsilon}(\mu) - \Psi_{1/\epsilon}(\mu_1)\big| = 0.$$

(c) Next we define the function $\Gamma : L_1^+(\Lambda_N) \to [0,\infty)$ by

$$\Gamma(f) = \int_{\Lambda_N} dx\big(1 - e^{-2\kappa f(x)}\big). \tag{2.2.78}$$

Below is the result from Lemma 7 from [BBH01].

Lemma 2.2.14. *For any $K > 0$,*

$$\lim_{\epsilon\downarrow 0} \sup_{\nu: \frac{d}{\epsilon} I_{\epsilon/d}^{(2)}(\nu) \leq K} \Big|\Gamma(\frac{d\nu}{dx}) - \Psi_{1/\epsilon}(\nu)\Big| = 0,$$

where $d\nu/dx$ is the density of ν with respect to Lebesgue measure. If ν do not have a density, then the supremum is infinite. Recall (2.2.24) and (2.2.76), the authors of [BBH01] also point out that, if $I_{\epsilon/d}(\nu) < \infty$, then $d\nu \ll dx$ because $\nu\otimes\pi_{\epsilon/d} \ll dx\otimes dy$.

2.2.4 Proof of Proposition 2.2.1

In this section, we collect the results from Section 2.2.1, Section 2.2.2 and Section 2.2.3 to complete the proof of Proposition 2.2.1.

Proof. For any $f : \mathbb{R}_+ \to \mathbb{R}$ bounded and continuous,

1. By Proposition 2.2.2, Proposition 2.2.4 and Varadhan's lemma,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n^{\frac{d-2}{d}}} \log \mathbb{E}\Big(\exp\big[n^{\frac{d-2}{d}} f(\tfrac{1}{n}\mathcal{R}_n)\big]\Big) \\
&\qquad = \lim_{\epsilon\downarrow 0} \lim_{n\to\infty} \frac{1}{n^{\frac{d-2}{d}}} \log \mathbb{E}\Bigg(\exp\Big[n^{\frac{d-2}{d}} f\Big(\mathbb{E}_{n,\epsilon}\big(\tfrac{1}{n}\mathcal{R}_n\big)\Big)\Big]\Bigg) \\
&\qquad = \lim_{\epsilon\downarrow 0} \sup_{\mu\in\mathcal{M}_1^+(\Lambda_N\times\Lambda_N)} \big\{ f\big(\Phi_{\frac{1}{\epsilon}}(\mu)\big) - \tfrac{1}{\epsilon} I^{(2)}_{\epsilon/d}(\mu)\big\}.
\end{aligned} \tag{2.2.79}$$

2. We will prove that

$$\lim_{\epsilon\downarrow 0} \sup_{\mu} \big\{ f\big(\Phi_{\frac{1}{\epsilon}}(\mu)\big) - \tfrac{1}{\epsilon} I^{(2)}_{\epsilon/d}(\mu)\big\} = \lim_{K\to\infty} \lim_{\epsilon\downarrow 0} \sup_{\mu:\frac{d}{\epsilon} I^{(2)}_{\epsilon/d}(\mu)\le K} \big\{ f\big(\Phi_{\frac{1}{\epsilon}}(\mu)\big) - \tfrac{1}{\epsilon} I^{(2)}_{\epsilon/d}(\mu)\big\}. \tag{2.2.80}$$

   Note that since f is bounded, we have

$$\sup_{\mu} f\big(\Phi_{\frac{1}{\epsilon}}(\mu)\big) - \tfrac{1}{\epsilon} I^{(2)}_{\epsilon/d}(\mu) \ge -\sup|f|.$$

   Now, we can see that any μ with $\frac{1}{\epsilon} I^{(2)}_{\epsilon/d}(\mu) > 2\sup|f|$ can be discounted. Therefore, by setting $K = 2d\sup|f|$, we get the equation.

3. By Lemma 2.2.13, Equation (2.2.76), Lemma 2.2.14, Lemma 2.2.12 and Equation

(2.2.75) respectively, we get:

$$\begin{aligned}
&\lim_{K\to\infty}\lim_{\epsilon\downarrow 0}\sup_{\mu:\frac{d}{\epsilon}I^{(2)}_{\epsilon/d}(\mu)\le K}\left\{f\big(\Phi_{\frac{1}{\epsilon}}(\mu)\big)-\tfrac{1}{\epsilon}I^{(2)}_{\epsilon/d}(\mu)\right\}\\
&\quad=\lim_{K\to\infty}\lim_{\epsilon\downarrow 0}\sup_{\mu:\frac{d}{\epsilon}I^{(2)}_{\epsilon/d}(\mu)\le K}\left\{f\big(\Psi_{\frac{1}{\epsilon}}(\mu_1)\big)-\tfrac{1}{\epsilon}I^{(2)}_{\epsilon/d}(\mu)\right\}\\
&\quad=\lim_{K\to\infty}\lim_{\epsilon\downarrow 0}\sup_{\nu:\frac{d}{\epsilon}I_{\epsilon/d}(\nu)\le K}\left\{f\big(\Psi_{\frac{1}{\epsilon}}(\nu)\big)-\tfrac{1}{\epsilon}I_{\epsilon/d}(\nu)\right\}\\
&\quad=\lim_{K\to\infty}\lim_{\epsilon\downarrow 0}\sup_{\nu:\frac{d}{\epsilon}I_{\epsilon/d}(\nu)\le K}\left\{f\left(\Gamma\Big(\frac{d\nu}{dx}\Big)\right)-\tfrac{1}{\epsilon}I_{\epsilon/d}(\nu)\right\}\\
&\quad=\sup_{\nu}\left\{f\left(\Gamma\Big(\frac{d\nu}{dx}\Big)\right)-\tfrac{1}{d}I(\nu)\right\}\\
&\quad=\sup_{\phi\in H^1(\Lambda_N):||\phi||_2^2=1}\left\{f\big(\Gamma(\phi^2)\big)-\frac{1}{2d}||\nabla\phi||_2^2\right\}. \qquad (2.2.81)
\end{aligned}$$

Combining (2.2.79), (2.2.80), (2.2.81) and recalling (2.2.78), we can see that the claim now follows from the inverse of Varadhan's lemma proved in Bryc [Bry90]. □

2.3 Proof of Theorem 1.2.10

In this Section we complete the proof of Theorem 1.2.10. This will be done by deriving the upper and lower bounds of the left hand side of (1.2.25) in Section 2.3.1 and Section 2.3.2 respectively.

First of all, we recall and rename Proposition 1.2.13 which will be used in both proofs of the upper and lower bounds.

Proposition 2.3.1. $\lim_{N\to\infty} I_N^\kappa(b) = I^\kappa(b)$ *for all* $b>0$ *where*

- $I_N^\kappa(b)$ *is given by the same formula as in (1.2.26) and (1.2.27) except that* $\mathbb{R}^d$ *is replaced by* Λ_N.
- I^κ *is the rate function defined in Theorem 1.2.10.*

The proof of Proposition 2.3.1 is omitted since it is exactly the same as the proof of Proposition 2 in Section 2.6 in [BBH01], where κ can be taken as arbitrary positive number.

2.3.1 The upper bound

Proof. We shall prove (1.2.28):

$$\limsup_{n\to\infty} \frac{1}{n^{\frac{d-2}{d}}} \log \mathbb{P}\big(\tfrac{1}{n} R_n \le b\big) \le -\tfrac{1}{d} I^{\kappa}(b).$$

Note that Proposition 2.2.1 implies the following :

Corollary 2.3.2. *Let $d \ge 3$. For every $b > 0$ and $N > 0$,*

$$\lim_{n\to\infty} \frac{1}{n^{\frac{d-2}{d}}} \log \mathbb{P}\big(\tfrac{1}{n} \mathcal{R}_n \le b\big) = -\tfrac{1}{d} I_N^{\kappa}(b).$$

Now, it is trivial that $\mathcal{R}_n \le R_n$ and therefore

$$\frac{1}{n^{\frac{d-2}{d}}} \log \mathbb{P}\big(\tfrac{1}{n} R_n \le b\big) \le \frac{1}{n^{\frac{d-2}{d}}} \log \mathbb{P}\big(\tfrac{1}{n} \mathcal{R}_n \le b\big),$$

for all $b > 0, N > 0$ and $n > 0$. As $n \to \infty$, by using Corollary 2.3.2 and Proposition 2.3.1, we now complete the proof of (1.2.28). □

2.3.2 The lower bound

Proof. We shall prove (1.2.29):

$$\liminf_{n\to\infty} \frac{1}{n^{\frac{d-2}{d}}} \log \mathbb{P}\big(\tfrac{1}{n} R_n \le b\big) \ge -\tfrac{1}{d} I^{\kappa}(b).$$

Firstly, we let $C_{Nn^{1/d}}(n)$ be the event that the random walk $(S_n : n = 1, 2, \ldots)$ does not hit the boundary $\partial\Lambda_{Nn^{1/d}}$ until time n. Then, clearly,

$$\mathbb{P}\big(\tfrac{1}{n} R_n \le b\big) \ge \mathbb{P}\big(C_{Nn^{1/d}}(n), \tfrac{1}{n} \mathcal{R}_n \le b\big). \tag{2.3.1}$$

Now, the right hand side involves the random walk on the torus with the restriction that the walk does not hit the boundary. Now, we repeat the arguments of Section 2.2 on the event $C_{Nn^{1/d}}(n)$, i.e. with zero boundary conditions instead of the periodic ones considered. Note that one major difference is that we can use the local central limit theorem, see (2.1.2), without requiring Lemma 2.1.2. The arguments of the proofs of Proposition 2.2.2 and Proposition 2.2.4, as well as Proposition 2.2.11 and the proof of Proposition 2.2.1 in Section 2.2.4, are still valid. Therefore, we get

$$\lim_{n\to\infty} \frac{1}{n^{\frac{d-2}{d}}} \log \mathbb{P}\big(\tfrac{1}{n} \mathcal{R}_n \le b \,|\, C_{Nn^{1/d}}(n)\big) = -\tfrac{1}{d} \tilde{I}_N^{\kappa}(b) \tag{2.3.2}$$

where $\tilde{I}_N^\kappa(b)$ is the same rate function as in (1.2.26) and (1.2.27), except that ϕ is satisfying supp $(\phi) \cap \partial\Lambda_N = \emptyset$.
Now, let $S_k = (S_k^{(1)}, \dots, S_k^{(d)})$. For the event $C_{Nn^{1/d}}(n)$,

$$\begin{aligned}
\mathbb{P}(C_{Nn^{1/d}}(n)) &= \mathbb{P}\big(\max_{1\le k\le n} n^{-\frac{1}{2}}||S_k||_\infty \le \tfrac{N}{2} n^{\frac{1}{d}-\frac{1}{2}}\big) \\
&= \mathbb{P}\big(\bigcap_{j=1}^{d} \{\max_{1\le k\le n} n^{-\frac{1}{2}}|S_k^{(j)}| \le \tfrac{N}{2} n^{\frac{2-d}{2d}}\}\big) \\
&\ge \mathbb{P}\big(\bigcap_{j=1}^{d} \{\max_{1\le k\le n} n^{-\frac{1}{2}}|\tilde{S}_k^{(j)}| \le \tfrac{N}{2} n^{\frac{2-d}{2d}}\}\big) \\
&= \Big(\mathbb{P}\big(\max_{1\le k\le n} n^{-\frac{1}{2}}|\tilde{S}_k^{(1)}| \le \tfrac{N}{2} n^{\frac{2-d}{2d}}\big)\Big)^d, \qquad (2.3.3)
\end{aligned}$$

where $\big(\tilde{S}_i^{(1)}\big)_{1\le i\le n}, \dots, \big(\tilde{S}_i^{(d)}\big)_{1\le i\le n}$ are independent copies of one-dimensional simple random walk, and we use the fact that

$$\max_{1\le k\le n} |S_k^{(j)}| \le \max_{1\le k\le n} |\tilde{S}_k^{(j)}|,$$

for all $j = 1, \dots, d$. By Theorem 2.13 in [Rév05], for any $\varepsilon > 0$ and large n,

$$\begin{aligned}
\mathbb{P}\big(\max_{1\le k\le n} n^{-\frac{1}{2}}|\tilde{S}_k^{(1)}| \le \tfrac{N}{2} n^{\frac{2-d}{2d}}\big) &\ge \frac{4(1-\varepsilon)}{\pi}\Big[\exp(-\frac{\pi^2}{2N^2}n^{\frac{d-2}{d}}) - \frac{1}{3}\exp(-\frac{9\pi^2}{2N^2}n^{\frac{d-2}{d}})\Big] \\
&\ge \frac{8(1-\varepsilon)}{3\pi}\exp\Big[-\frac{\pi^2}{2N^2}n^{\frac{d-2}{d}}\Big]. \qquad (2.3.4)
\end{aligned}$$

Hence, by (2.3.3) and (2.3.4),

$$\lim_{n\to\infty} \frac{1}{n^{\frac{d-2}{d}}} \log \mathbb{P}(C_{Nn^{1/d}}(n)) \ge -\frac{d\pi^2}{2N^2} =: -\lambda_N, \qquad (2.3.5)$$

from which we can see that $\lim_{N\to\infty} \lambda_N = 0$. Combining (2.3.1), (2.3.2) and (2.3.5), we can deduce that

$$\liminf_{n\to\infty} \frac{1}{n^{\frac{d-2}{d}}} \log \mathbb{P}\big(\tfrac{1}{n}R_n \le b\big) \ge -\tfrac{1}{d}\tilde{I}_N^\kappa(b) - \lambda_N \qquad \text{for all } N.$$

By the same type of argument as in the proof of Proposition 2.3.1, we can deduce that

$$\lim_{N\to\infty} \tilde{I}_N^\kappa(b) = I^\kappa(b).$$

Hence, let $N \to \infty$, we can deduce (1.2.29). Therefore, Theorem 1.2.10 follows. $\square$

Chapter 3

Large deviation for the intersections of independent ranges

The structure of this section is as follows: In Section 3.1, we recall the notations and standard compactification described earlier in Section 1.3.3. Then, we prove the large deviation on the number of intersection made by two independent random walks on a torus. This is an analogous result of Proposition 1.2.11. The main steps of the proof are similar to that in Proposition 1.2.11 which is done in Section 2.2. The proof will be divided into three main steps, and the structure is as described in Section 1.3.3. Then, we start the proof of our main result, Theorem 1.3.7, in Section 3.2 where we complete the lower bound proof of the theorem. Finally, we complete the proof of the upper bound in Section 3.3. The proof will be divided into five steps and the structure is described in Section 1.3.3.

3.1 LDP for the intersection of the ranges of random walks on the torus

In this section, we recall the notations described at the beginning of Section 1.1 and standard compactification described in Section 1.3.3. Then, we prove Proposition 1.3.9. The material of the proof is mainly borrowed from Section 2.2 where we prove the large deviation result for $\frac{1}{n}\mathcal{R}_n$. The result will be used to complete the proof of Theorem 1.3.7 in Section 3.2 and Section 3.3.

Recall that $(S_n^1 : n = 1, 2, \ldots)$ and $(S_n^2 : n = 1, 2, \ldots)$ be two independent random walks on $\mathbb{Z}^d$ with R_n^1 and R_n^2 the corresponding ranges of each random walks up to time n. Also, recall that J_n is the number of intersection sites made by the two random walks as described in (1.1.3). For $N \in \mathbb{N}$ even, recall that Λ_N is the torus of size $N > 0, \Lambda_N = [-\frac{N}{2}, \frac{N}{2})^d$ with periodic boundary conditions. In this section, we let the walks live on $\Lambda_{Nn^{\frac{1}{d}}} \cap \mathbb{Z}^d$ with N fixed. In a similar way as in Section 2.2, we denote $(\mathcal{S}_n^1 : n = 1, 2, \ldots)$ and $(\mathcal{S}_n^2 : n = 1, 2, \ldots)$ the corresponding random walks of $(S_i^1)_{1 \le i \le n}$ and $(S_i^2)_{1 \le i \le n}$ on $\Lambda_{Nn^{\frac{1}{d}}} \cap \mathbb{Z}^d$. Also, $\mathcal{R}_n^1$ and $\mathcal{R}_n^2$ represent, respectively, the number of lattice sites visited by the random walks on the torus up to time n. Moreover, we have

$$\mathcal{J}_n = \sharp\big\{\{\mathcal{S}_j^1\}_{1 \le j \le n} \cap \{\mathcal{S}_j^2\}_{1 \le j \le n}\big\} \tag{3.1.1}$$

to be the number of intersection sites made by the two random walks up to time n. We recall and rename Proposition 1.3.9 described in Section 1.3.3.

Proposition 3.1.1. *$\frac{1}{n}\mathcal{J}_n$ satisfies the large deviation principle on $\mathbb{R}_+$ with rate $n^{\frac{d-2}{d}}$ and with rate function $\frac{1}{d}\hat{L}_N^\kappa$ where*

$$\hat{L}_N^\kappa(b) = \inf_{\phi \in \hat{\Phi}_N^\kappa(b)} \Big[\int_{\Lambda_N} | \bigtriangledown \phi|^2(x) dx\Big], \tag{3.1.2}$$

where

$$\hat{\Phi}_N^\kappa(b) = \{\phi \in H^1(\Lambda_N) : \int_{\Lambda_N} \phi^2(x)dx = 1, \int_{\Lambda_N} \Big(1 - e^{-\kappa\phi^2(x)}\Big)^2 dx \ge b\}. \tag{3.1.3}$$

This result is an analogous of Proposition 1.2.11. The main steps of the proof will also be similar to that in Proposition 1.2.11. We therefore divide the proof into three main steps as described in Section 1.3.3. The proof follows closely Proposition 2 of [BBH04].

3.1.1 Approximation of $\frac{1}{n}\mathcal{J}_n$ by $\mathbb{E}_{n,\epsilon}^{(2)}\frac{1}{n}\mathcal{J}_n$

Similar to Section 2.2.1, we show that $\frac{1}{n}\mathcal{J}_n$ can be well approximated by its conditional expectation. Recall the skeleton walks described in (1.3.43): For $j = 1, 2$ and $\epsilon > 0$ fixed

$$\mathbb{S}_{n,\epsilon}^j = \{\mathcal{S}^j_{i\epsilon n^{\frac{2}{d}}}\}_{1 \le i \le \frac{1}{\epsilon} n^{\frac{d-2}{d}}}.$$

Also, recall that $\mathbb{E}_{n,\epsilon}^{(2)} = \mathbb{E}(\cdot | \mathbb{S}_{n,\epsilon}^1, \mathbb{S}_{n,\epsilon}^2)$ denotes the conditional expectation given $\mathbb{S}_{n,\epsilon}^1, \mathbb{S}_{n,\epsilon}^2$ and $\mathbb{P}_{n,\epsilon}^{(2)}$ denote the conditional probability given $\mathbb{S}_{n,\epsilon}^1, \mathbb{S}_{n,\epsilon}^2$. In this section, we show that $\mathcal{J}_n$ can be approximated by its conditional expectation given $\mathbb{S}_{n,\epsilon}^1, \mathbb{S}_{n,\epsilon}^2$.

Proposition 3.1.2. *For all $\delta > 0$,*

$$\lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}\frac{1}{n^{(d-2)/d}}\log\mathbb{P}\big(\tfrac{1}{n}|\mathcal{J}_n-\mathbb{E}^{(2)}_{n,\epsilon}\mathcal{J}_n|\ge\delta\big)=-\infty.$$

Proof. Set

$$\beta_k=\{S_i^k : i=1,\dots,n\},\qquad k=1,2,$$

to be the lattice sites visited by the k-th random walk. Our aim is to use Proposition 2.2.2 in Section 2.2.1 for the proof. Note that we can deduce Proposition 3.1.2 from the following equations: Firstly,

$$\lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}\frac{1}{n^{(d-2)/d}}\log\mathbb{P}\big(\tfrac{1}{n}\big|\mathcal{J}_n-\mathbb{E}(\mathcal{J}_n|\mathbb{S}^1_{n,\epsilon},\beta_2)\big|\ge\delta\big|\beta_2\big)=-\infty, \tag{3.1.4}$$

uniformly in the realisation of β_2. And, secondly

$$\lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}\frac{1}{n^{(d-2)/d}}\log\mathbb{P}\big(\tfrac{1}{n}|\mathbb{E}(\mathcal{J}_n|\mathbb{S}^1_{n,\epsilon},\beta_2)-\mathbb{E}(\mathcal{J}_n|\mathbb{S}^1_{n,\epsilon},\mathbb{S}^2_{n,\epsilon})\big|\ge\delta\big)=-\infty. \tag{3.1.5}$$

Combining (3.1.4) and (3.1.5), we get Proposition 3.1.2.

Proof of (3.1.4)

Note that the proof of (3.1.4) can be adapted from the proof of Proposition 2.2.2 as follows: In the single random walk case, we are interested in how many sites on $\mathbb{Z}^d$ are visited by the random walk. For the intersection problem, we are interested in how many elements in the set β_2 are visited by random walk $(\mathcal{S}_i^1)_{1\le i\le n}$, i.e. $\mathcal{J}_n$ can be written as

$$\frac{1}{n}\sharp\Big\{\Big\{\bigcup_{i=1}^{\frac{1}{\epsilon}n^{\frac{d-2}{d}}}\mathcal{W}_i^1\Big\}\cap\beta_2\Big\},$$

where $\mathcal{W}_i^1$ is defined similar to (2.2.5). Then, it is obvious to see that we extend the proof of Proposition 2.2.2 to this case, using Talagrand's concentration inequality in Lemma 2.2.3. In general, for any measurable set $D\subset\Lambda_{Nn^{1/d}}$, the function

$$\{\mathcal{W}_i^1\}_{1\le i\le\frac{1}{\epsilon}n^{\frac{d-2}{d}}}\mapsto\sharp\Big\{\Big\{\bigcup_{i=1}^{\frac{1}{\epsilon}n^{\frac{d-2}{d}}}\mathcal{W}_i^1\Big\}\cap D\Big\},$$

is Lipschitz-continuous in the sense of (2.2.15) uniformly in D. Hence, (3.1.4) follows.

Proof of (3.1.5)

By interchanging β_1 and β_2 in (3.1.4), we get

$$\lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}\frac{1}{n^{(d-2)/d}}\log\mathbb{P}\big(\tfrac{1}{n}\big|\mathcal{J}_n-\mathbb{E}(\mathcal{J}_n|\beta_1,\mathbb{S}^2_{n,\epsilon})\big|\geq\delta|\beta_1\big)=-\infty, \tag{3.1.6}$$

uniformly in the realisation of β_1. Note that (3.1.6) implies that

$$\lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}\frac{1}{n^{(d-2)/d}}\log\mathbb{P}\big(\tfrac{1}{n}\big|\mathbb{E}(\mathcal{J}_n|\mathbb{S}^1_{n,\epsilon},\beta_2)-\mathbb{E}(\mathcal{J}_n|\mathbb{S}^1_{n,\epsilon},\mathbb{S}^2_{n,\epsilon})\big|\geq\delta|\mathbb{S}^1_{n,\epsilon}\big)=-\infty, \tag{3.1.7}$$

uniformly in the realisation of $\mathbb{S}^1_{n,\epsilon}$. By averaging over $\mathbb{S}^1_{n,\epsilon}$ in (3.1.7), we finally get (3.1.5).

$\square$

3.1.2 The LDP for $\mathbb{E}^{(2)}_{n,\epsilon}\frac{1}{n}\mathcal{J}_n$

In this section, we prove the large deviation principle for $\mathbb{E}^{(2)}_{n,\epsilon}\frac{1}{n}\mathcal{J}_n$. Note that, similar to the set up in Section 2.2.2, we will do a scaling of the torus from $\Lambda_{Nn^{1/d}}$ to Λ_N to remove n-dependence. We also have an issue to have random walks live on a scaled grids, therefore we will make similar assumption as described at the beginning of Section 2.2.2: Unless stated otherwise, $\tilde{\mathcal{S}}_n=a$ will have the same meaning as $\tilde{\mathcal{S}}_n=\lfloor a\rfloor$ where $\lfloor a\rfloor=(\lfloor a_1\rfloor,\ldots,\lfloor a_d\rfloor)$ with $\lfloor a_i\rfloor$ is the biggest integer less than or equal to a_i. Also, recall that, for $k=1,2,\tilde{\mathcal{S}}^k_n$ is the corresponding position of the random walk $\mathcal{S}^k_n$ on Λ_N. Note that, the scaling does not effect $\mathcal{J}_n$.

We recall a similar empirical measure introduced in (1.3.44)

$$L_{k,n,\epsilon}=\epsilon n^{-\frac{d-2}{d}}\sum_{i=1}^{\frac{1}{\epsilon}n^{\frac{d-2}{d}}}\delta_{\left(n^{-1/d}\mathcal{S}^k_{(i-1)\epsilon n^{2/d}},\,n^{-1/d}\mathcal{S}^k_{i\epsilon n^{2/d}}\right)},\qquad k=1,2.$$

Also, recall the entropy function, $I^{(2)}_\epsilon:\mathcal{M}^+_1(\Lambda_N\times\Lambda_N)\to[0,\infty]$ defined in (2.2.24)

$$I^{(2)}_\epsilon(\mu)=\begin{cases}h(\mu|\mu_1\otimes\pi_\epsilon) & \text{if } \mu_1=\mu_2\\ \infty & \text{otherwise,}\end{cases} \tag{3.1.8}$$

where, as usual, $h(\cdot|\cdot)$ denotes relative entropy between measures, μ_1 and μ_2 are the two marginals of μ and $\pi_\epsilon(x,dy)=p^\pi_\epsilon(y-x)dy$ is the Brownian transition kernel on Λ_N defined in (2.2.25).

Next, for $\eta > 0$, let $\hat{\Phi}_\eta : \mathcal{M}_1^+(\Lambda_N \times \Lambda_N) \to [0, \infty)$ be the function

$$\hat{\Phi}_\eta(\mu_1, \mu_2) = \int_{\Lambda_N} dx \Big(1 - \exp\big[- 2\eta\kappa \int_{\Lambda_N \times \Lambda_N} \varphi_\epsilon\big(y - x, z - x\big)\mu_1(dy, dz)\big]\Big)$$
$$\Big(1 - \exp\big[- 2\eta\kappa \int_{\Lambda_N \times \Lambda_N} \varphi_\epsilon\big(y - x, z - x\big)\mu_2(dy, dz)\big]\Big), \quad (3.1.9)$$

where, we recall from (2.2.27) that,

$$\varphi_\epsilon(y, z) = \frac{\int_0^\epsilon ds p^\pi_{s/d}(-y) p^\pi_{(\epsilon - s)/d}(z)}{p^\pi_{\epsilon/d}(z - y)}. \quad (3.1.10)$$

Our main result in this section is the following:

Proposition 3.1.3. *$\mathbb{E}^{(2)}_{n,\epsilon} \frac{1}{n} \mathcal{J}_n$ satisfies a LDP on $\mathbb{R}^+$ with speed $n^{\frac{d-2}{d}}$ and rate function*

$$J_{\epsilon/d}(b) = \inf\Big\{ \frac{1}{\epsilon}\Big(I^{(2)}_{\epsilon/d}(\mu_1) + I^{(2)}_{\epsilon/d}(\mu_2)\Big) : \mu_1, \mu_2 \in \mathcal{M}_1^+(\Lambda_N \times \Lambda_N), \hat{\Phi}_{1/\epsilon}(\mu_1, \mu_2) = b \Big\}.$$

This is an analogous result to Proposition 2.2.4.

Proof. We follow the first two steps as in the proof of Proposition 2.2.4 in page 54. We approximate $\mathcal{J}_n$ by cutting small holes around the points $\mathcal{S}^1_{i\epsilon n^{2/d}}, \mathcal{S}^2_{i\epsilon n^{2/d}}$, where $1 \le i \le \frac{1}{\epsilon} n^{\frac{d-2}{d}}$. By the similar procedure as in (2.2.30), we get

$$\frac{1}{n}\Big|\mathcal{J}_n - \mathcal{J}_n^K\Big| \le \frac{2^d c_1 K^d}{\epsilon n^{2/d}}, \quad (3.1.11)$$

which tends to zero as $n \to \infty$ and therefore is negligible. Next, we recall the quantity defined in (2.2.31). For $y, z \in \Lambda_N$, define

$$\begin{aligned} q_{n,\epsilon}(y, z) &= P\big(\sigma \le \epsilon n^{\frac{2}{d}} | \tilde{\mathcal{S}}_0 = y, \tilde{\mathcal{S}}_{\epsilon n^{\frac{2}{d}}} = z\big) \\ &= \mathbb{P}\big(\sigma \le \epsilon n^{\frac{2}{d}} | \mathcal{S}_0 = y n^{\frac{1}{d}}, \mathcal{S}_{\epsilon n^{\frac{2}{d}}} = z n^{\frac{1}{d}}\big), \end{aligned}$$

where $\sigma = \min\{n : \mathcal{S}_n = 0\} = \min\{n : \tilde{\mathcal{S}}_n = 0\}$. Now, define $\mathcal{W}_i^{1,K}$ and $\mathcal{W}_i^{2,K}$ as similar to (2.2.28). By the similar way as in (2.2.32), we can write the conditional expectation as:

$$
\begin{aligned}
&\mathbb{E}^{(2)}_{n,\epsilon}\frac{1}{n}\mathcal{J}^K_n \\
&\quad= \frac{1}{n}\sum_{x\in\Lambda_{Nn^{\frac{1}{d}}}} \Big(1-\mathbb{P}^{(2)}_{n,\epsilon}\big(x\notin \bigcup_{i=1}^{\frac{1}{\epsilon}n^{\frac{d-2}{d}}} \mathcal{W}^{1,K}_i\big)\Big)\Big(1-\mathbb{P}^{(2)}_{n,\epsilon}\big(x\notin \bigcup_{i=1}^{\frac{1}{\epsilon}n^{\frac{d-2}{d}}} \mathcal{W}^{2,K}_i\big)\Big) \\
&\quad= \int_{\Lambda_N} dx\Bigg(1-\exp\bigg(\frac{1}{\epsilon}n^{\frac{d-2}{d}}\int_{\Lambda_N\times\Lambda_N} \\
&\qquad\qquad L_{1,n,\epsilon}(dy,dz)\log\big[1-q^{Kn^{-1/d}}_{n,\epsilon}(y-\lfloor xn^{\frac{1}{d}}\rfloor n^{-\frac{1}{d}}, z-\lfloor xn^{\frac{1}{d}}\rfloor n^{-\frac{1}{d}})\big]\bigg)\Bigg) \\
&\qquad\times\Bigg(1-\exp\bigg(\frac{1}{\epsilon}n^{\frac{d-2}{d}}\int_{\Lambda_N\times\Lambda_N} \\
&\qquad\qquad L_{2,n,\epsilon}(dy,dz)\log\big[1-q^{Kn^{-1/d}}_{n,\epsilon}(y-\lfloor xn^{\frac{1}{d}}\rfloor n^{-\frac{1}{d}}, z-\lfloor xn^{\frac{1}{d}}\rfloor n^{-\frac{1}{d}})\big]\bigg)\Bigg),
\end{aligned}
\tag{3.1.12}
$$

where for $\rho > 0$, we remind that $q^{\rho}_{n,\epsilon}(y,z) = q_{n,\epsilon}(y,z)$ if $y,z\notin B_\rho$, the centred ball of radius ρ, and zero otherwise.

The next step is the key part of the proof. This is to show that the difference between the number of intersections given their conditional expectation and the function of μ_1,μ_2 defined in (3.1.9) can be written as the sum of a function of each measure individually. This allows us to apply the result from Proposition 2.2.4 straightaway. This method was done in pp. 753 of [BBH04]. We repeat the method from the paper:

For $k=1,2$ and $x\in\Lambda_N$, we set

$$
\begin{aligned}
f_k(x) &:= \exp\bigg(\frac{1}{\epsilon}n^{\frac{d-2}{d}}\int_{\Lambda_N\times\Lambda_N} L_{k,n,\epsilon}(dy,dz) \\
&\qquad\qquad \log\big[1-q^{Kn^{-1/d}}_{n,\epsilon}(y-\lfloor xn^{\frac{1}{d}}\rfloor n^{-\frac{1}{d}}, z-\lfloor xn^{\frac{1}{d}}\rfloor n^{-\frac{1}{d}})\big]\bigg) \\
g_k(x) &:= \exp\bigg(-\frac{2\kappa}{\epsilon}\int_{\Lambda_N\times\Lambda_N}\varphi_\epsilon(y-x,z-x)L_{k,n,\epsilon}(dy,dz)\bigg).
\end{aligned}
\tag{3.1.13}
$$

Then, we can write:

$$\begin{aligned}
&\mathbb{E}^{(2)}_{n,\epsilon}\frac{1}{n}\mathcal{J}_n - \hat{\Phi}_{1/\epsilon}(L_{1,n,\epsilon}, L_{2,n,\epsilon})\\
&\quad= \int_{\Lambda_N} dx\big(1-f_1(x)\big)\big(1-f_2(x)\big) - \int_{\Lambda_N} dx\big(1-g_1(x)\big)\big(1-g_2(x)\big)\\
&\quad= \int_{\Lambda_N} dx\big(g_1(x)-f_1(x)\big)\big(1-f_2(x)\big) + \int_{\Lambda_N} dx\big(1-g_1(x)\big)\big(g_2(x)-f_2(x)\big).
\end{aligned} \tag{3.1.14}$$

Let $k=1,2$. Since $q^{Kn^{-1/d}}_{n,\epsilon}(\cdot)$ is probability of an event, we have $\log\big[1-q^{Kn^{-1/d}}_{n,\epsilon}(\cdot)\big]\leq 0$. This gives $f_k(x)$ is an exponential of a non-positive term. Hence, we have $|1-f_k(x)|\leq 1$. Similarly, since $\varphi_\epsilon(\cdot)$ is non-negative function, this implies that $g_k(x)$ is also an exponential of a non-positive term. Hence, $|1-g_k(x)|\leq 1$. Using these two facts along with (3.1.14), we get

$$\begin{aligned}
|\mathbb{E}^{(2)}_{n,\epsilon}\frac{1}{n}\mathcal{J}_n - &\hat{\Phi}_{1/\epsilon}(L_{1,n,\epsilon}, L_{2,n,\epsilon})|\\
&\leq \int_{\Lambda_N} dx|g_1(x)-f_1(x)| + \int_{\Lambda_N} dx|g_2(x)-f_2(x)|.
\end{aligned} \tag{3.1.15}$$

Therefore, we can do the approximations on $L_{1,n,\epsilon}$ and $L_{2,n,\epsilon}$ separately, which is exactly done in Section 2.2.2. Next, we need to show that the left hand side of (3.1.15) converges to zero as $n\to\infty$ follows by $\epsilon\downarrow 0$. We consider each term on the right hand side of (3.1.15) separately. Note that

$$\int_{\Lambda_N} dx|g_1(x)-f_1(x)| = \int_{\Lambda_N} dx\big|\big(1-f_1(x)\big)-\big(1-g_1(x)\big)\big|.$$

By recall $f_1(x)$ and $g_1(x)$ defined in (3.1.13) we get

$$\begin{aligned}
&\int_{\Lambda_N} dx\big|\big(1-f_1(x)\big)-\big(1-g_1(x)\big)\big|\\
&\quad= \int_{\Lambda_N} dx\Bigg|\Bigg(1-\exp\bigg(\frac{1}{\epsilon}n^{\frac{d-2}{d}}\int_{\Lambda_N\times\Lambda_N}\\
&\qquad\qquad L_{1,n,\epsilon}(dy,dz)\log\big[1-q^{Kn^{-1/d}}_{n,\epsilon}(y-\lfloor xn^{\frac{1}{d}}\rfloor n^{-\frac{1}{d}}, z-\lfloor xn^{\frac{1}{d}}\rfloor n^{-\frac{1}{d}})\big]\bigg)\Bigg)\\
&\qquad-\Bigg(1-\exp\bigg(-\frac{2\kappa}{\epsilon}\int_{\Lambda_N\times\Lambda_N}\varphi_\epsilon(y-x,z-x)L_{1,n,\epsilon}(dy,dz)\bigg)\Bigg)\Bigg|,
\end{aligned}$$

we can see that this is exactly the same as $|\frac{1}{n}\mathbb{E}_{n,\epsilon}\mathcal{R}^{1,K}_n - \Phi_{\infty,1/\epsilon,0}(L_{1,n,\epsilon})|$, which has already been studied in Section 2.2.2. Hence we can immediately apply the result from Section 2.2.2, namely (2.2.72), to deduce the large deviation principle for each term.

Note that, the similar arguments also apply for the second term on the right hand side of (3.1.15).

Finally, we can deduce the rate function in Proposition 3.1.3 since we have the sum of two explicit rate functions for each random walk. $\square$

3.1.3 The limit $\epsilon \downarrow 0$ and the proof of Proposition 3.1.1

The structure of this section is similar as what was done in Section 2.2.3 and Section 2.2.4. We first introduce more approximate functions. Then, we complete the proof of Proposition 3.1.3.

Set $\mu_1, \mu_2 \in \mathcal{M}_1(\Lambda_N)$,

$$\begin{aligned} \hat{\Psi}_{1/\epsilon}(\mu_1, \mu_2) = \int_{\Lambda_N} dx \Big(1 - \exp\big[-\frac{2\kappa}{\epsilon} \int_0^\epsilon ds \int_{\Lambda_N} p_s(x-y)\mu_1(dy)\big]\Big) \\ \times \Big(1 - \exp\big[-\frac{2\kappa}{\epsilon} \int_0^\epsilon ds \int_{\Lambda_N} p_s(x-y)\mu_2(dy)\big]\Big), \end{aligned} \tag{3.1.16}$$

and for $f_1, f_2 \in L_1^+(\Lambda_N)$,

$$\hat{\Gamma}(f_1, f_2) = \int_{\Lambda_N} dx \Big(1 - e^{-2\kappa f_1(x)}\Big)\Big(1 - e^{-2\kappa f_2(x)}\Big). \tag{3.1.17}$$

Also, recall (2.2.75) and (2.2.76) that I_ϵ is the rate function of the discrete-time Markov chain on Λ_N with Brownian transition kernel p_ϵ, i.e.,

$$I_\epsilon(\nu) = \inf(I_\epsilon^{(2)}(\mu) : \mu_1 = \nu) \tag{3.1.18}$$

Next, we finalise the proof of Proposition 3.1.1. This will be done by obtaining the limit when ϵ goes to zero.

Proof. By Proposition 3.1.2, Proposition 3.1.3 and Varadhan's lemma, for $f : \mathbb{R}_+ \to \mathbb{R}$ bounded and continuous, we get

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n^{\frac{d-2}{d}}} \log \mathbb{E}\Big(\exp\big[n^{\frac{d-2}{d}} f(\tfrac{1}{n}\mathcal{J}_n)\big]\Big) \\ = \lim_{\epsilon\downarrow 0} \sup_{\mu_1,\mu_2 \in \Lambda_N\times\Lambda_N} \Big\{ f\big(\hat{\Phi}_{1/\epsilon}(\mu_1,\mu_2)\big) - \tfrac{1}{\epsilon}\big(I_{\epsilon/d}^{(2)}(\mu_1) + I_{\epsilon/d}^{(2)}(\mu_2)\big)\Big\}. \end{aligned} \tag{3.1.19}$$

Then, we repeat the approximation arguments similar to Section 2.2.4 and we get from

(3.1.19) that

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n^{\frac{d-2}{d}}} \log \mathbb{E}\Big(\exp\big[n^{\frac{d-2}{d}} f(\tfrac{1}{n}\mathcal{J}_n)\big]\Big) \\
&\qquad = \lim_{K\to\infty} \lim_{\epsilon\downarrow 0} \sup_{\nu_1,\nu_2:\frac{d}{\epsilon}I_{\epsilon/d}(\nu_1)\le K, \frac{d}{\epsilon}I_{\epsilon/d}(\nu_2)\le K} \Big\{ f\big(\Psi_{1/\epsilon}(\nu_1,\nu_2)\big) - \tfrac{1}{\epsilon}\big(I_{\epsilon/d}(\nu_1) + I_{\epsilon/d}(\nu_2)\big)\Big\} \\
&\qquad = \sup_{i=1,2:\phi\in H^1(\Lambda_N), ||\phi_i||_2^2=1} \Big\{ f\big(\hat\Gamma(\phi_1^2,\phi_2^2)\big) - \tfrac{1}{2d}\big(||\nabla\phi_1||_2^2 + ||\nabla\phi_2||_2^2\big)\Big\}. \qquad (3.1.20)
\end{aligned}$$

Using Bryc's lemma [Bry90], we see from (3.1.20) that $\frac{1}{n}\mathcal{J}_n$ satisfies the large deviation principle with speed $n^{(d-2)/d}$ and with rate function

$$\begin{aligned}
\hat{L}(b) &= \inf\Big\{ \frac{1}{2d}\big(||\nabla\phi_1||_2^2 + ||\nabla\phi_2||_2^2\big) : \\
&\qquad ||\phi_1||_2^2 = ||\phi_2||_2^2 = 1, \int_{\Lambda_N} dx \Big(1 - e^{-2\kappa\phi_1^2(x)}\Big)\Big(1 - e^{-2\kappa\phi_2^2(x)}\Big) \ge b\Big\} \\
&= \inf\Big\{ \frac{1}{d}||\nabla\phi||_2^2 : ||\phi||_2^2 = 1, \int_{\Lambda_N} dx \Big(1 - e^{-\kappa\phi^2(x)}\Big)^2 \ge b\Big\}. \qquad (3.1.21)
\end{aligned}$$

Note that the variational problem reduces to the diagonal $\phi_1 = \phi_2$ in the last equality by (3.1.22) and (3.1.23) below. We set $\phi^2 = \frac{1}{2}(\phi_1^2 + \phi_2^2)$ and note that

$$\phi = \sqrt{\tfrac{1}{2}(\phi_1^2 + \phi_2^2)} \le \frac{1}{\sqrt{2}}(\phi_1 + \phi_2).$$

By using the relations in (1.3.34) and (1.3.35), we get

$$|\nabla\phi|^2 \le \tfrac{1}{2}|\nabla\phi_1|_2^2 + \tfrac{1}{2}|\nabla\phi_2|_2^2. \qquad (3.1.22)$$

Next, we use that fact that $x \mapsto -e^{2\kappa\phi^2(x)}$ is concave to show that

$$\begin{aligned}
\big(1 - e^{-2\kappa\phi^2(x)}\big)^2 &= 1 - 2e^{-2\kappa\phi^2(x)} + e^{-4\kappa\phi^2(x)} \\
&\ge 1 - \Big(e^{-2\kappa\phi_1^2(x)} + e^{-2\kappa\phi_2^2(x)}\Big) + e^{-2\kappa\phi_1^2(x)}e^{-2\kappa\phi_2^2(x)} \\
&\ge \big(1 - e^{-2\kappa\phi_1^2(x)}\big)\big(1 - e^{-2\kappa\phi_2^2(x)}\big). \qquad (3.1.23)
\end{aligned}$$

Note that (3.1.21) is the required rate function for Proposition 3.1.1, and this completes the proof. □

3.2 The lower bound in Theorem 1.3.7

In this section, we complete the proof of (1.3.39):

$$\liminf_{n\to\infty} \frac{1}{n^{(d-2)/d}} \log \mathbb{P}(\tfrac{1}{n} J_n \geq a) \geq -\tfrac{1}{d} L^{\kappa}(a), \tag{3.2.1}$$

where $L^{\kappa}(a)$ is as given in (1.3.26) and (1.3.27). The proof is similar to the proof in Section 2.3.2 and the idea of the proof follows from Section 2.2 of [BBH04].

Proof. Firstly, we let $C^2_{Nn^{1/d}}(n)$ be the event that both of the two random walks do not hit the boundary of $\Big[-\frac{N}{2} n^{\frac{1}{d}}, \frac{N}{2} n^{\frac{1}{d}} \Big)^d$, hence stay in the torus of size $Nn^{1/d}$, until time cn. Clearly,

$$\mathbb{P}(\tfrac{1}{n} J_n \geq a) \geq \mathbb{P}\big(\tfrac{1}{n} \mathcal{J}_n \geq a, C^2_{Nn^{1/d}}(n)\big) \tag{3.2.2}$$

Now, similar to Section 2.3.2, we repeat the argument that led to Proposition 3.1.1, with the restriction on the event $C^2_{Nn^{1/d}}(n)$. We then get

$$\lim_{n\to\infty} \frac{1}{n^{(d-2)/d}} \log \mathbb{P}\big(\tfrac{1}{n} \mathcal{J}_n \geq a | C^2_{Nn^{1/d}}(n)\big) = -\tfrac{1}{d} \tilde{L}^{\kappa}_N(a), \tag{3.2.3}$$

where $\tilde{L}^{\kappa}_N(a)$ is the same rate function as in (3.1.2) and (3.1.3), except that ϕ is satisfying the extra restriction supp $(\phi) \cap \partial\big\{ -(\frac{N}{2} n^{\frac{1}{d}}, \frac{N}{2} n^{\frac{1}{d}})^d \big\} = \emptyset$. Next, we recall from Section 2.3.2 that $C_{Nn^{1/d}}(n)$ is the event that a random walk does not hit the boundary of $\Big[-\frac{N}{2} n^{\frac{1}{d}}, \frac{N}{2} n^{\frac{1}{d}} \Big)^d$. By a similar calculation as in Section 2.3.2, we get

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n^{\frac{d-2}{d}}} \log \mathbb{P}\big(C^2_{Nn^{1/d}}(n)\big) &= \lim_{n\to\infty} \frac{1}{n^{\frac{d-2}{d}}} \log \mathbb{P}\big(C_{Nn^{1/d}}(n)\big)^2 \\ &\geq -\frac{d\pi^2}{N} = -2\lambda_N, \end{aligned} \tag{3.2.4}$$

where the inequality comes from (2.3.5) and λ_N is also defined in (2.3.5). We also remind that $\lim_{N\to\infty} \lambda_N = 0$. Next, combine (3.2.2)-(3.2.4), we get

$$\liminf_{n\to\infty} \frac{1}{n^{\frac{d-2}{d}}} \log \mathbb{P}\big(\tfrac{1}{n} J_n \leq a\big) \geq -\tfrac{1}{d} \tilde{L}^{\kappa}_N(a) - 2\lambda_N \qquad \text{for all } N > 0. \tag{3.2.5}$$

Now, let $N \to \infty$ and note that by Proposition 1.3.10, we get

$$\lim_{N\to\infty} \tilde{L}^{\kappa}_N(a) = L^{\kappa}(a). \tag{3.2.6}$$

We will not prove Proposition 1.3.10 but the can be done in a similar way as in Proposition 1.2.13. This completes the proof of (1.3.39). □

3.3 The upper bound in Theorem 1.3.7

In this section, we prove (1.3.38). For the ease of reading, we translate the equation to the following proposition:

Proposition 3.3.1. *Let $d \geq 3$. Then, for every $a > 0$,*

$$\limsup_{n\to\infty} \frac{1}{n^{(d-2)/d}} \log \mathbb{P}(\tfrac{1}{n} J_n \geq a) \leq -\tfrac{1}{d} L^{\kappa}(a), \tag{3.3.1}$$

where $L^{\kappa}(a)$ is as given in (1.3.26) and (1.3.27).

The proof of Proposition 3.3.1 is divided into five steps. The structure of the proof is as described in Section 1.3.3.

3.3.1 Preliminaries

We divide this section into four main steps. In the first step, we introduce the models which will be used through out the proof. In Step 2 and Step 3, we define important quantities of the proof. Finally, to prepare for the proof in later sections, we introduce and prove a few results of the quantities we described in first three steps.

(1) We will make a partition of $\mathbb{Z}^d$ by the following: Assume $N > 0$ and $0 < \eta < N/2$. Define $\Theta_{Nn^{1/d}}$ to be a d-dimensional box of side-length $Nn^{1/d}$ i.e.

$$\Theta_{Nn^{1/d}} = \Big[-\frac{1}{2}Nn^{1/d}, -\frac{1}{2}Nn^{1/d}\Big)^d.$$

Then, this will partition $\mathbb{Z}^d$ into $Nn^{1/d}$-boxes as:

$$\mathbb{Z}^d = \bigcup_{z\in\mathbb{Z}^d} \Theta_{Nn^{1/d}}(z), \tag{3.3.2}$$

where $\Theta_{Nn^{1/d}}(z) = \Theta_{Nn^{1/d}} + zNn^{1/d}$. This **box partition** will be used throughout the section.

Now, we may also partition $\mathbb{Z}^d$ into d-dimensional slices, by the following: For each direction $k \in \{1, \ldots, d\}$, we can separate $\mathbb{Z}^d$ into d-dimension slices, $t^{(k)}$ with width $\eta n^{1/d}$ in direction k and infinite width in the other directions, i.e.

$$t^{(k)}_m = \Big\{(z_1, \ldots, z_d) \in \mathbb{Z}^d : z_k \in \big[-\tfrac{\eta}{2}n^{1/d} + m\eta n^{1/d}, \tfrac{\eta}{2}n^{1/d} + m\eta n^{1/d}\big)\Big\}, \quad m \in \mathbb{Z}. \tag{3.3.3}$$

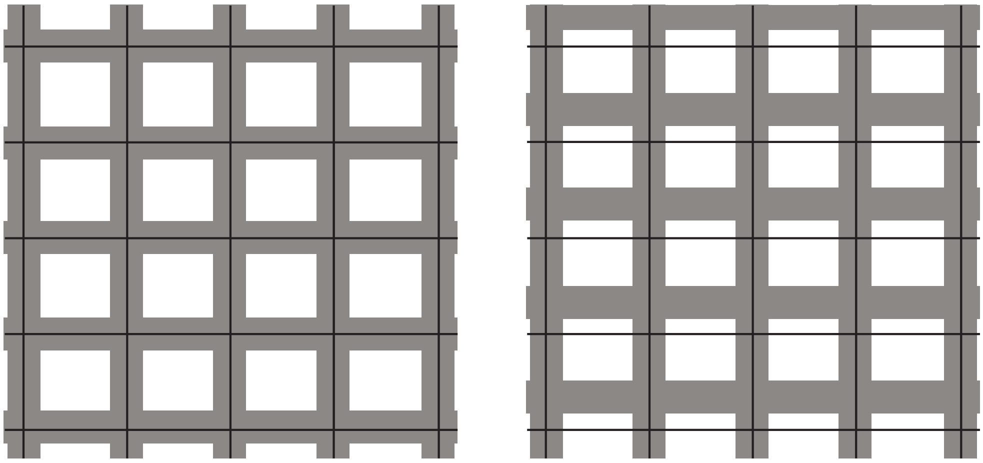

Figure 3-1: Partition of $\mathbb{Z}^2$ into boxes of sidelength $Nn^{1/d}$ and copies of $Q_{\eta,N,n}$ defined in (3.3.4) and (3.3.5). The shaded area represents the $\frac{1}{2}\eta n^{1/d}$-neighborhood of the faces of the boxes. The picture on the left shows the copy $Q^{(0,0)}_{\eta,N,n}$, while the copy on the right represent $Q^{(0,1)}_{\eta,N,n}$.

For example in direction 1, we can have slices as:

$$\ldots, \Big[-\frac{3\eta}{2}n^{1/d}, -\frac{\eta}{2}n^{1/d}\Big) \times \mathbb{Z}^{d-1}, \Big[-\frac{\eta}{2}n^{1/d}, \frac{\eta}{2}n^{1/d}\Big) \times \mathbb{Z}^{d-1}, \Big[\frac{\eta}{2}n^{1/d}, \frac{3\eta}{2}n^{1/d}\Big) \times \mathbb{Z}^{d-1}, \ldots.$$

These slices will be important tools later on.

(2) We introduce copies of the box partition. Let $Q_{\eta,N,n}$ denote the $\frac{1}{2}\eta n^{1/d}$-neighborhood of the faces of the boxes, i.e.

$$Q_{\eta,N,n} = \bigcup_{z\in\mathbb{Z}^d} \Big(\big(\Theta_{Nn^{1/d}} \setminus \Theta_{(N-\eta)n^{1/d}}\big) + zNn^{1/d}\Big) \tag{3.3.4}$$

Assume N/η is an *even* integer. If we shift $Q_{\eta,N,n}$ by $\eta n^{1/d}$ for N/η times in each of the d directions and in every possible combinations we obtain $(N/\eta)^d$ copies of $Q_{\eta,N,n}$:

$$Q^x_{\eta,N,n} = Q_{\eta,N,n} + x\eta n^{\frac{1}{d}}, \qquad x = (x_1,\ldots,x_d) \in \big\{0,\ldots,\tfrac{N}{\eta}-1\big\}^d, \tag{3.3.5}$$

where $Q^x_{\eta,N,n}$ is the shift, which was made by shifting $Q_{\eta,N,n}$ by $x_k\eta n^{1/d}$ in each direction k. Each point of $\mathbb{Z}^d$ is contained in exactly $(N/\eta)^d - (N/\eta-1)^d$ copies. See Figure 3-1, for example.

Now, we can see that each copy of $Q_{\eta,N,n}$ can be formed by forming the union of particular slices in (3.3.3). However, the most important remark is that each slice is contained in exactly $(N/\eta)^{d-1}$ copies.

(3) We are going to look at how often the random walks $(S_i^1)_{1\leq i\leq n}$ and $(S_i^2)_{1\leq i\leq n}$ cross *the slices* in direction $k \in \{1,\ldots,d\}$ of width $\eta n^{1/d}$. Define

$$\mathcal{B}^{(k)} = \Big\{(z_1,\ldots,z_d) \in \mathbb{Z}^d : z_k = \Big(\frac{\eta}{2}+\eta a\Big)n^{1/d} \text{ for some } a \in \mathbb{Z}\Big\}, \tag{3.3.6}$$

to be set of points on the boundary hyperplanes *between* slices on direction k. Also, for $x = (x_1,\ldots,x_d) \in \mathcal{B}^{(k)}$, define

$$\mathcal{B}_x^{(k)} = \big\{(z_1,\ldots,z_d) \in \mathbb{Z}^d : z_k = x_k\big\} \tag{3.3.7}$$

to be set of points on the boundary hyperplane that contains x.

Obviously, $\mathcal{B}_x^{(k)} \subset \mathcal{B}^{(k)}$. Also, for $i = 1, 2$, define

$$\begin{aligned} T_1^{i,(k)} &= \min\{m > 0 : S_m^i \in \mathcal{B}^{(k)}\} \\ T_2^{i,(k)} &= \min\{m > T_1^{i,(k)} : S_m^i \in \mathcal{B}^{(k)}, \mathcal{B}^{(k)}_{S_m^i} \neq \mathcal{B}^{(k)}_{S^i_{T_1^{i,(k)}}}\}, \\ &\vdots \qquad\qquad \vdots \\ T_j^{i,(k)} &= \min\{m > T_{j-1}^{i,(k)} : S_m^i \in \mathcal{B}^{(k)}, \mathcal{B}^{(k)}_{S_m^i} \neq \mathcal{B}^{(k)}_{S^i_{T_{j-1}^{i,(k)}}}\}, \end{aligned} \tag{3.3.8}$$

to be the steps taken to cross the slices in direction k (of width $\eta n^{1/d}$) of the random walks. Now, we are going to define the crossings on the slices of width $\eta n^{1/d}$ in direction k. Let

$$\Upsilon_j^{i,(k)} = \max\{T_j^{i,(k)} \leq m < T_{j+1}^{i,(k)} : S_m^i \in \mathcal{B}^{(k)}_{S^i_{T_j^{i,(k)}}}\}, \tag{3.3.9}$$

to be the last time that the random walk i hit the current boundary hyperplane $\mathcal{B}^{(k)}_{S^i_{T_j^{i,(k)}}}$, before hitting the new boundary hyperplane $\mathcal{B}^{(k)}_{S^i_{T_{j+1}^{i,(k)}}}$. Now, we can see that the path

$$\{S_m^i\}_{\Upsilon_j^{i,(k)} \leq m \leq T_{j+1}^{i,(k)}}$$

lies fully inside a slice of length $\eta n^{1/d}$ in direction k. We call this path the *crossing* of the slice. Now, define $C_n^{(k)}(\eta)$ to be the total number of crossings made by two random walks up to time n in direction k. It is clear that

$$C_n^{(k)}(\eta) = \max\{j : T_{j-1}^{1,(k)} \leq n\} + \max\{j : T_{j-1}^{2,(k)} \leq n\}. \tag{3.3.10}$$

Finally, define

$$C_n(\eta) = \sum_{k=1}^{d} C_n^{(k)}(\eta) \tag{3.3.11}$$

to be the total number of crossings made by the random walks. Now, we introduce the *central hyperplanes* on the slices of width $\eta n^{1/d}$ on direction k, which lie at the centre of these slices, i.e.

$$\mathcal{H}^{(k)} = \Big\{(z_1, \ldots, z_d) \in \mathbb{Z}^d : z_k = a\eta n^{1/d} \text{ for some } a \in \mathbb{Z}\Big\}. \tag{3.3.12}$$

Obviously, for each crossing, the random walk will hit the central hyperplane of the slice. Now, we define the *entrance time* of a crossing to be the *first* time when the crossing hits the central hyperplane. Similarly, we define the *exit time* of a crossing to be the *last* time where the crossing hits the central hyperplane. The reason to introduce the central hyperplanes is that, we will do reflections on the path of the walks on these central hyperplanes.
Next, for a slice $\mathrm{H}_1^{(k)}$ of width $\eta n^{1/d}$ and its central hyperplane, $\mathcal{H}_1^{(k)}$, define

- A *good excursion* of $\mathcal{H}_1^{(k)}$ to be the path of a random walk that starts from an exit time of $\mathcal{H}_1^{(k)}$ of any crossing of $\mathrm{H}_1^{(k)}$ and ends at the entrance time of $\mathcal{H}_1^{(k)}$ of the next crossing on $\mathrm{H}_1^{(k)}$.
- A *bad path* of $\mathcal{H}_1^{(k)}$ to be the path of a random walk that starts from the entrance time of $\mathcal{H}_1^{(k)}$ of any crossing of $\mathrm{H}_1^{(k)}$ and ends at the exit time of the same crossing.
- An *exit excursion* of $\mathcal{H}_1^{(k)}$ to be the path of a random walk that starts from the last time that the random walk hits $\mathcal{H}_1^{(k)}$.
- An *entrance excursion* of $\mathcal{H}_1^{(k)}$ to be the path of a random walk that starts from time zero and ends at the first entrance time the walk hits $\mathcal{H}_1^{(k)}$.

In order to do the reflection on $\mathcal{H}_1^{(k)}$, we only reflect some of the good excursions, exit excursions and entrance excursions of $\mathcal{H}_1^{(k)}$, leaving all bad paths unreflected. Which excursions that are actually reflected will become clear in Reflection argument 3.3.3.

(4) At a later step of the proof, we are going to do the reflection of the paths in various hyperplanes in order to move them inside a large time-dependent box. We now introduce the lemma which will be needed for the estimates. Define the event

$$\mathcal{O}_n = \Big\{S_j^k \in \big[-n, n\big]^d, 0 \le j \le n, k = 1, 2\Big\} \tag{3.3.13}$$

Lemma 3.3.2. *(a)* $\lim_{n\to\infty} \dfrac{1}{n^{(d-2)/d}} \log \mathbb{P}([\mathcal{O}_n]^c) = -\infty$

(b) $\lim_{n\to\infty} \frac{1}{n^{(d-2)/d}} \log \mathbb{P}\big(\frac{1}{n} J_n > 2\kappa\big) = -\infty.$

(c) For every $M > 0$,

$$\limsup_{\eta\to\infty} \limsup_{n\to\infty} \frac{1}{n^{(d-2)/d}} \log \mathbb{P}\Big(C_n(\eta) > \frac{dM}{\eta} n^{\frac{d-2}{d}}\Big) = -C(M), \tag{3.3.14}$$

with $\lim_{M\to\infty} C(M) = \infty$.

We abbreviate the last two events as:

$$\mathcal{V}_n = \big\{\tfrac{1}{n} J_n \le 2\kappa\big\}, \tag{3.3.15}$$

$$\mathcal{C}_{n,M,\eta} = \big\{C_n(\eta) \le \frac{dM}{\eta} n^{\frac{d-2}{d}}\big\}. \tag{3.3.16}$$

The lemma implies the following: **(i)** Until time n, the random walks can not travel further than the distance n **(ii)** the number of intersection points cannot be too large, and **(iii)** the total number of crossings in (3.3.11) cannot be too large.

Proof. **(a)** This is trivial since the random walks can not escape from the box $\big[-n, n\big]^d$. This gives $\mathbb{P}([\mathcal{O}_n]^c) = 0$ and hence Lemma 3.3.2 (a).

(b) Note that $J_n \le R_n^1$. Using Kesten and Hamana's result in Theorem 1.2.6, we have

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(R_n^1 \ge 2\kappa n) = -\psi(2\kappa), \tag{3.3.17}$$

where $\psi(2\kappa)$ is positive and finite. Next, since $n \gg n^{(d-2)/d}$, it can be deduced that

$$\lim_{n\to\infty} \frac{1}{n^{(d-2)/d}} \log \mathbb{P}(R_n^1 \ge 2\kappa n) = -\infty. \tag{3.3.18}$$

Therefore, we can deduce Lemma 3.3.2 (b).

(c) Since

$$\mathbb{P}\Big(C_n(\eta) > \frac{dM}{\eta} n^{\frac{d-2}{d}}\Big) \le d\mathbb{P}\Big(C_n^1(\eta) > \frac{M}{\eta} n^{\frac{d-2}{d}}\Big), \tag{3.3.19}$$

it is enough to estimate the η-crossings perpendicular to direction 1. For $i = 1, 2$, let $\tilde{T}_k^i = T_k^i - T_{k-1}^i$ with $T_0 = 0$. Then, $\tilde{T}_1, \tilde{T}_2, \ldots$ denote the independent and identically[1] distributed crossing time of the slices. Since, for both random walks, all the crossings

[1] By (3.3.8), the distribution of $\tilde{T}_1$ is different from $\tilde{T}_2, \tilde{T}_3, \ldots$. However, we can deal with this easily, and the rest of the proof remains valid.

must occur before time n, we have

$$\mathbb{P}\Big(C_n^{(1)}(\eta) > \frac{M}{\eta} n^{\frac{d-2}{d}}\Big) \le 2\mathbb{P}\Big(\sum_{i=1}^{\frac{M}{2\eta}n^{\frac{d-2}{d}}} \tilde{T}_i^{(1)} < n\Big). \tag{3.3.20}$$

Now, let $\tilde{\tau}_1, \tilde{\tau}_2, \ldots$ denote the independent and identically distributed crossing time taken by the *one-dimensional* random walk to cross the one-dimensional slice of width $\eta n^{1/d}$. Obviously,

$$\mathbb{P}\Big(\sum_{i=1}^{\frac{M}{2\eta}n^{\frac{d-2}{d}}} \tilde{T}_i < n\Big) \le \mathbb{P}\Big(\sum_{i=1}^{\frac{M}{2\eta}n^{\frac{d-2}{d}}} \tilde{\tau}_i < n\Big). \tag{3.3.21}$$

Now, note that the event $\Big\{\sum_{i=1}^{\frac{M}{2\eta}n^{\frac{d-2}{d}}} \tilde{\tau}_i < n\Big\}$ implies that at least half of one-dimensional η-crossing time is less than $\frac{4\eta}{M}n^{2/d}$. Since all the time are independent and identically distributed, we have,

$$\mathbb{P}\Big(\sum_{i=1}^{\frac{M}{2\eta}n^{\frac{d-2}{d}}} \tilde{\tau}_i < n\Big) \le \xi\Big(\mathbb{P}(\tilde{\tau}_1 < \tfrac{4\eta}{M}n^{2/d})\Big)^{\frac{M}{4\eta}n^{\frac{d-2}{d}}}, \tag{3.3.22}$$

where $\xi = \binom{\lfloor \frac{M}{2\eta}n^{\frac{d-2}{d}} \rfloor}{\lfloor \frac{M}{4\eta}n^{\frac{d-2}{d}} \rfloor}$ is the number of permutation to choose $\lfloor \frac{M}{2\eta}n^{\frac{d-2}{d}} \rfloor$ events out of $\lfloor \frac{M}{4\eta}n^{\frac{d-2}{d}} \rfloor$. Note that, by Stirling's formula,

$$\log \xi = \frac{M}{2\eta} n^{\frac{d-2}{d}} \log 2 + o(\tfrac{M}{2\eta}n^{\frac{d-2}{d}}). \tag{3.3.23}$$

Now, let $\mathcal{M}_k = \max_{1 \le i \le k} |S_i^1|$. From (3.3.22), we can deduce that

$$\begin{aligned}\mathbb{P}\big(\tilde{\tau}_1 < \tfrac{4\eta}{M}n^{2/d}\big) &= \mathbb{P}\big(\mathcal{M}_{\frac{4\eta}{M}n^{2/d}} > \eta n^{1/d}\big)\\ &= \mathbb{P}\Big(\frac{1}{\sqrt{\frac{4\eta}{M}n^{2/d}}}\mathcal{M}_{\frac{4\eta}{M}n^{2/d}} > \frac{\sqrt{M\eta}}{2}\Big).\end{aligned} \tag{3.3.24}$$

Now, by Theorem 2.13 from [Rév05] we have, for any $\varepsilon > 0$,

$$\mathbb{P}\Big(\frac{1}{\sqrt{\frac{4\eta}{M}n^{2/d}}}\mathcal{M}_{\frac{4\eta}{M}n^{2/d}} > \frac{\sqrt{M\eta}}{2}\Big) \le (1+\varepsilon)\frac{8}{2\pi M\eta}\exp\Big(-\frac{M\eta}{8}\Big). \tag{3.3.25}$$

Hence, from (3.3.19) - (3.3.25) we can deduce that

$$\mathbb{P}\Big(C_n(\eta) > \frac{dM}{\eta} n^{\frac{d-2}{d}}\Big) \le 2\xi d \Big[(1+\varepsilon)\frac{8}{\sqrt{2\pi M\eta}} \exp\Big(-\frac{M\eta}{8}\Big)\Big]^{\frac{M}{4\eta} n^{\frac{d-2}{d}}}$$
$$= 2\xi d \exp\Big(-\frac{M^2}{32} n^{\frac{d-2}{d}}\Big) \Big[(1+\varepsilon)\frac{8}{\sqrt{2\pi M\eta}}\Big]^{\frac{M}{4\eta} n^{\frac{d-2}{d}}}. \quad (3.3.26)$$

Therefore, by (3.3.23) and (3.3.26),

$$\limsup_{n\to\infty} \frac{1}{n^{(d-2)/d}} \log \mathbb{P}\Big(C_n(\eta) > \frac{dM}{\eta} n^{\frac{d-2}{d}}\Big)$$
$$\le \frac{M}{2\eta}\log 2 - \frac{M^2}{32} + \frac{M}{4\eta}\log\Big[(1+\varepsilon)\frac{8}{\sqrt{2\pi M\eta}}\Big] + o\Big(\frac{M}{2\eta}\Big),$$

and therefore,

$$\limsup_{\eta\to\infty} \frac{1}{n^{(d-2)/d}} \limsup_{n\to\infty} \log \mathbb{P}\Big(C_n(\eta) > \frac{dM}{\eta} n^{\frac{d-2}{d}}\Big) \le -\frac{M^2}{32}.$$

Hence, we get the claim in (3.3.14) with $C(M) = \dfrac{M^2}{32}$. $\square$

3.3.2 Counting the intersections

We start analysing $Q^x_{\eta,N,n}$ introduced in (3.3.5) and describe the set up to complete the proof in Section 3.3.5.

(a) For $x \in \big\{0, \dots, \frac{N}{\eta} - 1\big\}^d$, define

$$C_n(Q^x_{\eta,N,n}) = \sum_{i=1}^{2}\sum_{k=1}^{d} \sum_{l=0}^{\max\{j: T^{i,(k)}_{j-1} \le n\}} \mathbf{1}\{(S^i_j)_{T^{i,(k)}_l \le j \le T^{i,(k)}_{l+1}} \subset Q^x_{\eta,N,n}\} \quad (3.3.27)$$

to be the number of crossings in $Q^x_{\eta,N,n}$ up to time n, and

$$J_n(Q^x_{\eta,N,n}) = \sharp\Big\{\{S^1_i\}_{1\le i\le n} \cap \{S^2_i\}_{1\le i\le n} \cap Q^x_{\eta,N,n}\Big\}, \quad (3.3.28)$$

to be the number of intersection points in $Q^x_{\eta,N,n}$ up to time n. Now, by summing the total number of crossings for all copies of $Q_{\eta,N,n}$, each slice will be used exactly

$(N/\eta)^{d-1}$ times. Hence, on the event $\mathcal{C}_{n,M,\eta} \cap \mathcal{V}_n$, we have

$$\sum_{x\in\{0,\ldots,N/\eta-1\}^d} C_n(Q^x_{\eta,N,n}) \leq \frac{dMn^{\frac{d-2}{d}}}{\eta}\Big(\frac{N}{\eta}\Big)^{d-1}.$$
$$\sum_{x\in\{0,\ldots,N/\eta-1\}^d} \tfrac{1}{n}J_n(Q^x_{\eta,N,n}) \leq 2\kappa\Big(\frac{N}{\eta}\Big)^{d-1}. \tag{3.3.29}$$

Note that there exists a shift $X \in \{0,\ldots,\frac{N}{\eta}-1\}^d$ such that

$$C_n(Q^X_{\eta,N,n}) \leq \tfrac{2dM}{N} n^{\frac{d-2}{d}}, \tag{3.3.30}$$
$$\tfrac{1}{n}J_n(Q^X_{\eta,N,n}) \leq 4\kappa\tfrac{\eta}{N}. \tag{3.3.31}$$

Our aim now is to use a reflection procedure introduced in [BBH04] in order to control the random walks, and these two bounds will play crucial roles later. Next, we pick $\eta=\sqrt{N}$ and $M=\log N$and use the fact that for large N, both the number of crossings and the number of intersection points in $Q^X_{\eta,N,n}$ are small. This fact will allow us to control both the entropy associated with the reflections and the change in the number of intersection caused by the reflections.

(b) Before we describe the reflection procedure, we need some set up. Recall (3.3.5), let $x^X_{\sqrt{N},N,n}$ denotes the shift that $Q^X_{\sqrt{N},N,n}$ is obtained from $Q_{\sqrt{N},N,n}$. For $z\in\mathbb{Z}^d$, we define

$$\tfrac{1}{n}J^X_{n,N}(z) = \tfrac{1}{n}\sharp\big\{\{S^1_j\}_{1\leq j\leq n}\cap\{S^2_j\}_{1\leq j\leq n}\cap\Theta^X_{Nn^{1/d}}(z)\big\},$$
$$\tfrac{1}{n}J^X_{n,\sqrt{N},N,\mathrm{out}}(z) = \tfrac{1}{n}\sharp\Big\{\{S^1_j\}_{1\leq j\leq n}\cap\{S^2_j\}_{1\leq j\leq n}\cap Q^X_{\sqrt{N},N,n}(z)\Big\},$$
$$\tfrac{1}{n}J^X_{n,\sqrt{N},N,\mathrm{in}}(z) = \tfrac{1}{n}\sharp\big\{\{S^1_j\}_{1\leq j\leq n}\cap\{S^2_j\}_{1\leq j\leq n}\cap[\Theta^X_{Nn^{1/d}}(z)\setminus Q^X_{\sqrt{N},N,n}(z)]\big\}, \tag{3.3.32}$$

where

$$\Theta^X_{Nn^{1/d}}(z) = \Theta_{Nn^{1/d}} + zNn^{1/d} + x^X_{\sqrt{N},N,n}$$
$$Q^X_{\sqrt{N},N,n}(z) = [\Theta_{Nn^{1/d}}\setminus\Theta_{(N-\sqrt{N})n^{1/d}}] + zNn^{1/d} + x^X_{\sqrt{N},N,n}. \tag{3.3.33}$$

Next, define

$$\mathcal{Z}^X_{\epsilon,N} = \Big\{z\in\mathbb{Z}^d : \tfrac{1}{n}R^{1,X}_{n,N}(z) > \epsilon \text{ or } \tfrac{1}{n}R^{2,X}_{n,N}(z) > \epsilon\Big\}, \tag{3.3.34}$$

to be the set of *popular boxes*, where

$$R^{k,X}_{n,N}(z) = \sharp\Big\{\{S^k_j\}_{1\leq j\leq n}\cap(\Theta^X_{Nn^{1/d}}(z))\Big\}, \qquad k=1,2. \tag{3.3.35}$$

Also, we define $Z^X_{\epsilon,N} := \sharp\{\mathcal{Z}^X_{\epsilon,N}\}$, and define the event

$$\mathcal{R}^{(2)}_n = \left\{\tfrac{1}{n}R^1_n \le 2\kappa, \tfrac{1}{n}R^2_n \le 2\kappa\right\}, \tag{3.3.36}$$

where R^1_n and R^2_n is the number of distinct sites on $\mathbb{Z}^d$ visited by each random walk up to time n. Note that, trivially, $\mathcal{R}^{(2)}_n \subset \mathcal{V}_n$. Moreover, by (3.3.34) and (3.3.36), on the event $\mathcal{R}^{(2)}_n$ we have

$$\sharp\{\mathcal{Z}^X_{\epsilon,N}\} \le 4\kappa/\epsilon. \tag{3.3.37}$$

Also, by (3.3.18) we get

$$\lim_{n\to\infty} \frac{1}{n^{\frac{d-2}{d}}} \log \mathbb{P}\big([\mathcal{R}^{(2)}_n]^c\big) = -\infty. \tag{3.3.38}$$

(c) Now, we described the reflection procedure introduced in [BBH04]:

Reflection argument 3.3.3.

We start by a **labelling procedure**:

L1. We will deal with the reflection for each direction $k \in \{1, \dots, d\}$ separately. For each direction k, we will partition $\mathbb{Z}^d$ into slices $\mathbf{T}^{(k)}_m$ of width $Nn^{1/d}$, i.e. for $m \in \mathbb{Z}$,

$$\begin{aligned}\mathbf{T}^{(k)}_m := \Big\{&(z_1, \dots, z_d) \in \mathbb{Z}^d : \\ &z_k \in \Big[-\tfrac{N}{2}n^{1/d} + mNn^{1/d} + x^X_{\sqrt{N},N,n}, \tfrac{N}{2}n^{1/d} + mNn^{1/d} + x^X_{\sqrt{N},N,n}\Big)\Big\}.\end{aligned} \tag{3.3.39}$$

L2. We consider the collection of the popular boxes $Z^X_{\epsilon,N}$ in (3.3.34). From now on, we will only consider the slices that contain at least one popular box (we will call these slices the *popular slices*). Assume there are R such slices. Note that $R \le Z^X_{\epsilon,N}$. Now, we will label the popular slices by $H^{(k)}_1, \dots, H^{(k)}_R$ to be the popular slices evaluated from the left to the right.

Next, we define $d_1 Nn^{1/d}, \dots, d_{R-1}Nn^{1/d}$ to be the distances between the successive popular slices, i.e. if $H^{(k)}_1$ is connected to $H^{(k)}_2$, then $d_1 = 0$.

The reason that we introduce slices $\mathbf{T}^{(k)}_m$ of width $Nn^{1/d}$ is that each slice of $\mathbf{T}^{(k)}_m$ corresponds to $\bigcup_{\{z:z_k=m\}} \Theta^X_{Nn^{1/d}}(z)$ in direction k. Also, the unions of $\frac{1}{2}\eta$-neighborhood of the faces of the boxes make slices of width $\eta n^{1/d}$. Moreover, the central hyperplanes of the slices of width $\eta n^{1/d}$, see (3.3.12), will play the role of boundary hyperplanes between successive slices $\mathbf{T}_m^{(k)}$. See Figure 3-1.

Now, we are ready to do the **reflecting procedure** via the following explanations:

R1. We first consider d_1. If $d_1 \geq 1$, define the *central hyperplane* $\mathcal{H}_1^{(k)}$ uniquely by the following properties:

- $\mathcal{H}_1^{(k)}$ is amongst the central hyperplanes of slices of width $\eta n^{1/d}$, located between successive slices $\mathbf{T}_m^{(k)}$, that lie between $H_1^{(k)}$ and $H_2^{(k)}$.
- Reflecting on $\mathcal{H}_1^{(k)}$, the slice $H_2^{(k)}$ lands to the *left* of $H_1^{(k)}$ at a distance either 0 or N (depending whether d_1 is odd, respectively, even).

We then do the reflection on $\mathcal{H}_1^{(k)}$. If $d_1 = 0$, then we do not reflect the walks.

R2. To do the reflection, we only reflect on those good excursions, exit excursions and entrance excursions (see Page 93) that lie fully on the right of $\mathcal{H}_1^{(k)}$. We do not reflect bad paths.

R3. The effect of the first reflecting procedure (R1) is that slices $H_1^{(k)}$ and $H_2^{(k)}$ fall inside a slice of side-length $3Nn^{1/d}$, no matter whether we do the reflection or not.

R4. Now, we repeat the reflecting procedure with d_2. If $d_2 \geq 3$, we define the central hyperplane $\mathcal{H}_2^{(k)}$ uniquely by the following properties:

- $\mathcal{H}_2^{(k)}$ is amongst the central hyperplanes of slices of width $\eta n^{1/d}$, located between successive slices $\mathbf{T}_m^{(k)}$, that lie between $H_2^{(k)}$ and $H_3^{(k)}$.
- Reflecting on $\mathcal{H}_2^{(k)}$, the slice $H_2^{(k)}$ lands to the *right(left depending on whether there is a reflection on Step R1 or not)* of the slice of width $3Nn^{1/d}$ that contains $H_1^{(k)}$ and $H_2^{(k)}$(see R3), at a distance either $0, N$ or $2N$.

We then do the path reflection similar to R2 on $\mathcal{H}_2^{(k)}$. Note that if we make reflection from the first reflection procedure on R1, then we look for good excursions, exit excursions and entrance excursions that lies fully on the left of $\mathcal{H}_2^{(k)}$. If we do not reflect from the first reflection procedure, we looks for good excursions, exit excursions and entrance excursions on the right of $\mathcal{H}_2^{(k)}$. If $d_2 \leq 2$, then we do not reflect the walks.

R5. The effect of the second reflecting procedure is that the slice $H_1^{(k)}, H_2^{(k)}$ and $H_3^{(k)}$ fall inside a slice of side-length $6Nn^{1/d}$.

R6. Repeat the arguments for $d_3, \ldots, d_{R-1}$ i.e. compare whether $d_i \geq 3 \times 2^{i-2}$(do reflect) or $d_i \leq 3 \times 2^{i-2} - 1$(do not reflect).

R7. After all reflections have been made in direction 1, we repeat the label procedure and reflection procedure for direction $2, \ldots, k$.

It is clear by Figure 3-1 that the path reflection must be made at the central hyperplanes, in order to control the entropy in $\bigcup_{z\in\mathbb{Z}^d}\left(\Theta^X_{Nn^{1/d}}(z)\setminus Q^X_{\sqrt{N},N,n}(z)\right)$, the volume defined in (3.3.33).

The example in Figure 3-3 shows the *global picture* for a reflection procedure in $\mathbb{Z}^2$, while Figure 3-2 shows the *local picture* of what happens to a path when the reflection at the central hyperplanes has been made.

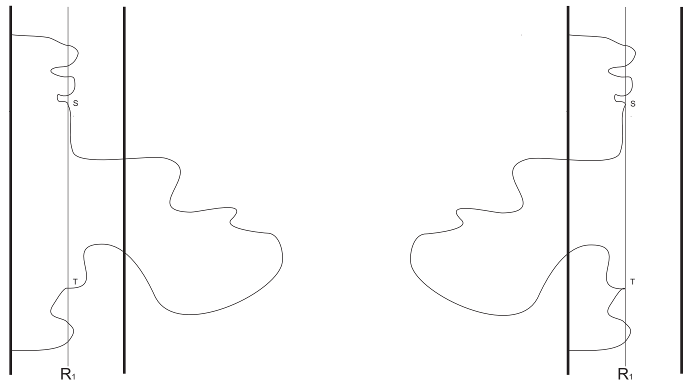

Figure 3-2: To reflect the random walk on the right of the central hyperplane R_1, we only reflect a good excursion that lies fully on the right of R_1, which is the path of random walks from an exit time S to the next entrance time T.

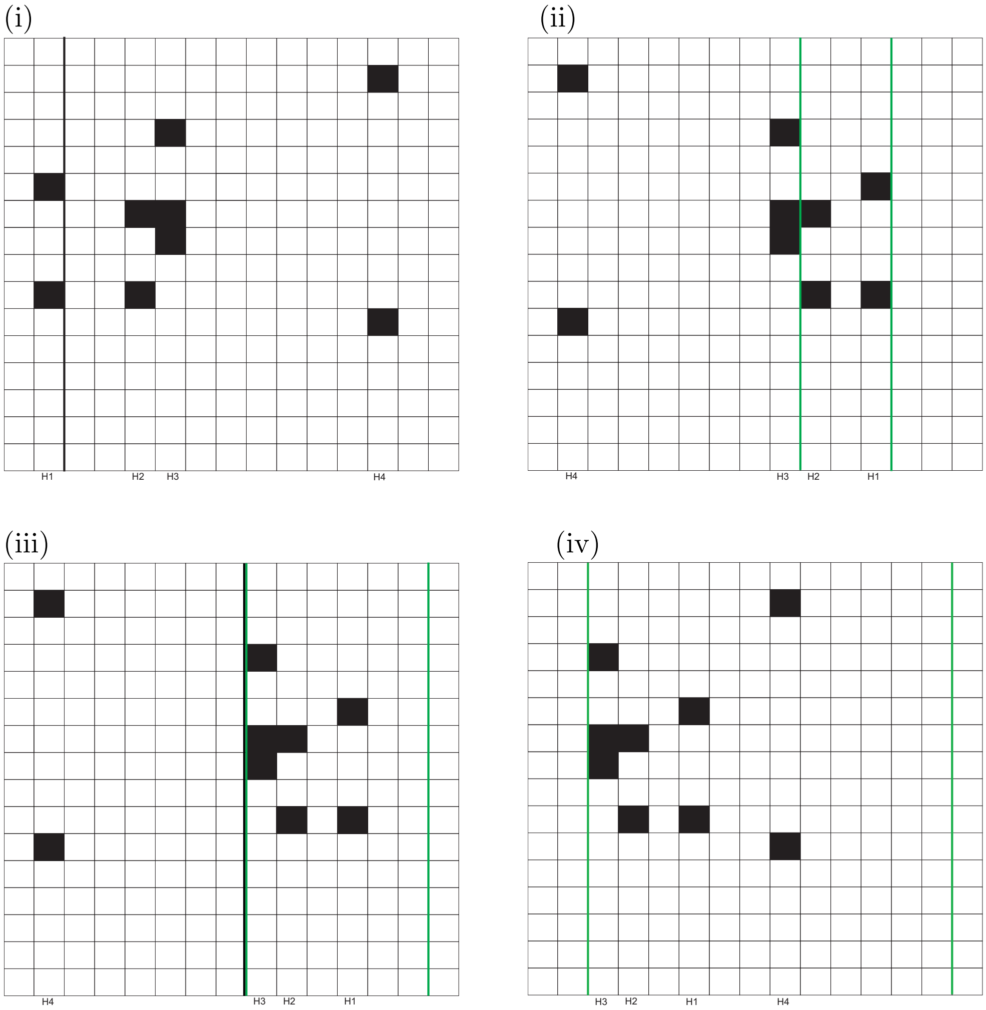

Figure 3-3: Reflection procedure on $\mathbb{Z}^2$ in direction 1 from top-left, top-right, bottom-left and bottom-right respectively: The popular boxes are represented by the black boxes. In direction 1, we have such 4 popular slices labelled H1, ..., H4 from left to right respectively. We can see that $d_1 = 2, d_2 = 0, d_3 = 6$ (i) Since $d_1 \geq 1$, we make a reflection on the bold hyperplane. (ii) The reflection makes all the popular slices move to the left of H1 and the new distance between H1 and H2 is 1. This makes H1 and H2 lie inside a slice of width $3Nn^{1/d}$, represented by the green boundaries, in direction 1. (iii) Since $d_2 \leq 3$ we do not make a reflection and H1, H2 and H3 lie inside a slice of width $6Nn^{1/d}$. Next, since $d_3 \geq 6$, we do a reflection on the bold hyperplane. (iv) The reflections made H1, H2, H3 and H4 lie inside a slice of width $12Nn^{1/d}$ in direction 1. After doing the similar procedure in direction 2, all the popular boxes will lie inside a $12Nn^{1/d}$ box.

(d) We end the section by introduce two results from Reflection argument 3.3.3.

Proposition 3.3.4. *For $N \geq 1$ fixed and $\epsilon, \delta > 0$,*

(a) By Reflection argument 3.3.3, the number of reflections made in the hyperplanes of $Q^X_{\sqrt{N},N,n}$ is at most $\sharp\{Z^X_{\epsilon,N}\} - 1$. After all the reflections have been made, all the intersection sets end up in disjoint boxes of sidelength $Nn^{1/d}$ inside a large box of sidelength $2^{Z^X_{\epsilon,N}} Nn^{1/d}$. Therefore, by projecting the reflected random walks on $\Lambda_{2^{Z^X_{\epsilon,N}} Nn^{1/d}}$ this will not affect the intersection $\sum_{z \in \mathcal{Z}^X_{\epsilon,N}} \frac{1}{n} J^X_{n,\sqrt{N},N,in}(z)$.

(b) Let $\mathcal{R}$ denotes the reflection transformation from Reflection argument 3.3.3 and $\tilde{P}$ the path measure for the two random walks defined by $\tilde{P}(A) = P(\mathcal{R}^{-1}A)$ where A is the set of paths of random walks. On the event, $\mathcal{O}_n \cap \mathcal{C}_{n,\log N,\sqrt{N}} \cap \mathcal{R}^{(2)}_n$, the cost of doing the reflections is at most $\exp[\gamma_N n^{\frac{d-2}{d}} + O(\log n)]$ as $n \to \infty$, with $\lim_{N\to\infty} \gamma_N = 0$, i.e.,

$$d\tilde{P}/dP \leq \exp[\gamma_N n^{(d-2)/d} + O(\log n)].$$

Proposition 3.3.5. *There exists an N_0 such that for every $0 < \epsilon \leq 1$ and $\delta > 0$,*

$$\begin{aligned}\limsup_{n\to\infty} \sup_{N \geq N_0} \frac{1}{n^{(d-2)/d}} \log \mathbb{P}\bigg(\Big\{\frac{1}{n} \sum_{z \in \mathbb{Z}^d \setminus \mathcal{Z}^X_{\epsilon,N}} J^X_{n,N} > \delta\Big\} \cup \Big\{\frac{1}{n} \sum_{z \in \mathbb{Z}^d} J^X_{n,\sqrt{N},N,out} > \delta\Big\}\bigg) \\ \leq -K(\epsilon, \delta), \qquad (3.3.40)\end{aligned}$$

with $\lim_{\epsilon \downarrow 0} K(\epsilon, \delta) = \infty$ for any $\delta > 0$.

Note that (3.3.40) imply, by the complement of the event, that

$$0 \leq \tfrac{1}{n} J_n - \tfrac{1}{n} \sum_{z \in \mathcal{Z}^X_{\epsilon,N}} J^X_{n,\sqrt{N},N,\mathrm{in}}(z) \leq 2\delta. \qquad (3.3.41)$$

Note that the sum in (3.3.41) is invariant under the Reflection argument 3.3.3 and therefore the estimate in (3.3.41) implies that most of the intersection points are unaffected after the reflections have been made.

3.3.3 Proof of Proposition 3.3.4

We proceed the proof of Proposition 3.3.4.

Proof. **(a)** After we consider the reflection procedure of all slices, we get all the R slices fit inside a slice of sidelength $3 \times 2^{R-2} Nn^{1/d}$ which is less than $\leq 2^R Nn^{1/d}$. After the reflection procedure has been made in direction $2, \ldots, k$, all the popular boxes fit inside a box of size $2^{Z^X_{\epsilon,N}} N n^{1/d}$. Note that by making reflections at the central hyperplanes, this make no effect on the number of intersections made in $\bigcup_{z\in\mathbb{Z}^d}\big(\Theta^X_{Nn^{1/d}}(z)\setminus Q^X_{\sqrt{N},N,n}(z)\big)$, the volume defined in (3.3.33).

(b) Note that the cost of adapting the reflections is bounded, restricted on the event $\mathcal{O}_n \cap \mathcal{C}_{n,\log N,\sqrt{N}} \cap \mathcal{R}^{(2)}_n$. This comes from the product of the three contributions:

- From the crossings of the random walks: By considering that each crossing defined in (3.3.11) has two possiblilities, to reflect or not to reflect. Also, by (3.3.11) and (3.3.16) with $M = \log N$ and $\eta = \sqrt{N}$, on the event $\mathcal{C}_{n,\log N,\sqrt{N}}$ the total number of crossings of the two random walks is bounded above by $d\frac{\log N}{\sqrt{N}} n^{\frac{d-2}{d}}$ from Lemma 3.3.2 (c). Therefore, this contributes at most $2^{d\frac{\log N}{\sqrt{N}} n^{\frac{d-2}{d}}}$ for the cost of applying the reflections.
- On the event $\mathcal{O}_n$ defined in (3.3.13) the number of central hyperplanes available for the reflection is bounded above by $\left(\frac{2n}{n^{1/d}}\right)^d$. Also, on the event $\mathcal{R}^{(2)}_n$ defined in (3.3.36) and from (3.3.37), the total number of reflections is bounded above by $|Z^X_{\epsilon,N}| \le 4\kappa/\epsilon$. Hence, this contributes at most $\left(\frac{2n}{n^{1/d}}\right)^{4d\kappa/\epsilon}$.
- The total number of shifted copies of $Q_{\sqrt{N},N}$ available defined in (3.3.4) is $\left(\frac{N}{\sqrt{N}}\right)^d$.

Therefore, by combining these three contributions, we get

$$\begin{aligned}\frac{d\tilde{P}}{dP} &\le 2^{d\frac{\log N}{\sqrt{N}} n^{\frac{d-2}{d}}}\left(\frac{2n}{n^{1/d}}\right)^{4d\kappa/\epsilon}\left(\frac{N}{\sqrt{N}}\right)^d \\ &= \exp\left[\log\left(2^{d\frac{\log N}{\sqrt{N}} n^{\frac{d-2}{d}}}\left(2n^{(d-1)/d}\right)^{4d\kappa/\epsilon} N^{d/2}\right)\right] \\ &= \exp\left[n^{\frac{d-2}{d}}\left(d\frac{\log N}{\sqrt{N}}\log 2\right) + \frac{4(d-1)\kappa}{\epsilon}\log n + \log\left(2^{(4d\kappa)/\epsilon} N^{d/2}\right)\right]. \end{aligned} \tag{3.3.42}$$

Hence, by setting $\gamma_N = d\frac{\log N}{\sqrt{N}}\log 2$, which gives $\lim_{N\to\infty}\gamma_N = 0$ and noting that

$$\exp\left[\tfrac{4(d-1)\kappa}{\epsilon}\log n + \log\left(2^{(4d\kappa)/\epsilon}N^{d/2}\right)\right] = \exp[O(\log n)],$$

we can deduce Proposition 3.3.4 (ii). □

3.3.4 Proof of Proposition 3.3.5

We proceed the proof of Proposition 3.3.5. The proof is divided into two steps.

Proof. **(1)** Firstly, the event

$$\Big\{\tfrac{1}{n}\sum_{z\in\mathbb{Z}^d} J^X_{n,\sqrt{N},N,\mathrm{out}} > \delta\Big\}$$

from (3.3.40) can be simplified. Note that $N \geq N_0 = (4\kappa/\delta)^2$ because of (3.3.31) with $\eta = \sqrt{N}$ and $M = \log N$ (recall that $\frac{1}{n}J_n(Q^X_{\sqrt{N},N}) = \frac{1}{n}\sum_{z\in\mathbb{Z}^d} J^X_{n,\sqrt{N},N,\mathrm{out}}(z)$). Thus, we only need to show that there exists an N_0 such that for every $0 < \epsilon \leq 1$ and $\delta > 0$,

$$\limsup_{n\to\infty}\sup_{N\geq N_0}\frac{1}{n^{(d-2)/d}}\log\mathbb{P}\bigg(\Big\{\frac{1}{n}\sum_{z\in\mathbb{Z}^d\setminus Z^X_{\epsilon,N}} J^X_{n,N} > \delta\Big\}\bigg) \leq -K(\epsilon,\delta), \tag{3.3.43}$$

with $\lim_{\epsilon\downarrow 0} K(\epsilon,\delta) = \infty$ for any $\delta > 0$. To do this, for $N \geq 1$ and $\epsilon > 0$, let

$$\mathcal{A}^{(n)}_{\epsilon,N} = \Big\{A \subset \mathbb{Z}^d : \inf_{x\in\mathbb{Z}^d}\sup_{z\in\mathbb{Z}^d}\tfrac{1}{n}\sharp\{(A+x)\cap\Theta_{Nn^{1/d}}(z)\} \leq \epsilon\Big\}. \tag{3.3.44}$$

Note that the class of sets $\mathcal{A}^{(n)}_{\epsilon,N}$ is closed under translations. Also, its elements become more sparse as $\epsilon \downarrow 0$. We prove Proposition 3.3.5 via the following lemma:

Lemma 3.3.6. *For every $0 < \epsilon \leq 1$ and $\delta > 0$,*

$$\lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}\frac{1}{n^{(d-2)/d}}\log\sup_{N\geq 1}\sup_{A\in\mathcal{A}^{(n)}_{\epsilon,N}}\mathbb{P}\big(\tfrac{1}{n}\sharp\big\{A\cap\{S_j\}_{1\leq j\leq n}\big\} \geq \delta\big) = -K(\epsilon,\delta), \tag{3.3.45}$$

with $\lim_{\epsilon\downarrow 0} K(\epsilon,\delta) = \infty$ for any $\delta > 0$.

We will complete the proof of the lemma later. Now, we finish the proof of Proposition 3.3.5. Note that Lemma 3.3.6 implies Proposition 3.3.5 as follows: Consider the random set

$$A^* = \bigcup_{\{z\in\mathbb{Z}^d : \frac{1}{n}\sharp\{\{S^1_j\}_{1\leq j\leq n}\cap\Theta^X_{Nn^{1/d}}(z)\}\leq\epsilon\}}\big\{\{S^1_j\}_{1\leq j\leq n}\cap\Theta^X_{Nn^{1/d}}(z)\big\}. \tag{3.3.46}$$

We can see that, $A^* \in \mathcal{A}^{(n)}_{\epsilon,N}$. Now, recall (3.3.32)-(3.3.34),

$$\begin{aligned}\frac{1}{n}\sum_{z\in\mathbb{Z}^d\setminus Z^X_{\epsilon,N}} J^X_{n,N}(z) &= \frac{1}{n}\sum_{z\in\mathbb{Z}^d\setminus Z^X_{\epsilon,N}}\sharp\Big\{\{S^1_j\}_{1\leq j\leq n}\cap\{S^2_j\}_{1\leq j\leq n}\cap\Theta^X_{Nn^{1/d}}(z)\Big\}\\ &\leq \frac{1}{n}\sum_{z\in\mathbb{Z}^d}\sharp\Big\{A^*\cap\{S^2_j\}_{1\leq j\leq n}\cap\Theta^X_{Nn^{1/d}}(z)\Big\}\\ &= \frac{1}{n}\sharp\Big\{A^*\cap\{S^2_j\}_{1\leq j\leq n}\Big\}.\end{aligned} \tag{3.3.47}$$

Therefore,

$$\mathbb{P}\Big(\frac{1}{n}\sum_{z\in\mathbb{Z}^d\setminus Z^X_{\epsilon,N}} J^X_{n,N}(z)>\delta\Big)\le \sup_{A\in\mathcal{A}^{(n)}_{\epsilon,N}}\mathbb{P}\Big(\tfrac{1}{n}\sharp\big\{A\cap\{S^2_j\}_{1\le j\le n}\big\}>\delta\Big). \tag{3.3.48}$$

By (3.3.48) along with Lemma 3.3.6 implies (3.3.43), and this completes the proof of Proposition 3.3.5. □

(2) Next, we prove Lemma 3.3.6.

Proof of Lemma 3.3.6. **(a)** We will show that

$$\lim_{\epsilon\downarrow 0}\limsup_{n\to\infty}\frac{1}{n^{(d-2)/d}}\log\sup_{N\ge 1}\sup_{A\in\mathcal{A}^{(n)}_{\epsilon,N}}\mathbb{E}\Big(\exp\Big[\epsilon^{-1/3d}n^{-\frac{2}{d}}\sharp\big\{A\cap\{S_j\}_{1\le j\le n}\big\}\Big]\Big)=0. \tag{3.3.49}$$

Now, using (3.3.49) together with Chebyshev's inequality that

$$\begin{aligned}\mathbb{P}\Big(&\tfrac{1}{n}\sharp\big\{A\cap\{S^2_j\}_{1\le j\le n}\big\}>\delta\Big)\\ &\le \exp\Big[-\delta\epsilon^{-\frac{1}{3d}}n^{\frac{d-2}{d}}\Big]\mathbb{E}\Big(\exp\Big[\epsilon^{-1/3d}n^{-\frac{2}{d}}\sharp\big\{A\cap\{S^2_j\}_{1\le j\le n}\big\}\Big]\Big),\end{aligned} \tag{3.3.50}$$

will imply Lemma 3.3.6.

(b) To prove (3.3.49), we use the subadditivity property of $s\to\frac{1}{n}\sharp\big\{A\cap s\big\}$ as follows:

$$\begin{aligned}&\sup_{A\in\mathcal{A}^{(n)}_{\epsilon,N}}\mathbb{E}\Big(\exp\Big[\epsilon^{-\frac{1}{3d}}n^{-\frac{2}{d}}\sharp\big\{A\cap\{S_j\}_{1\le j\le n}\big\}\Big]\Big)\\ &\quad\le \sup_{A\in\mathcal{A}^{(n)}_{\epsilon,N}}\mathbb{E}\Big(\exp\Big[\epsilon^{-\frac{1}{3d}}n^{-\frac{2}{d}}\sum_{k=1}^{\epsilon^{-\frac{1}{d}}n^{\frac{d-2}{d}}}\sharp\big\{A\cap\{S_j\}_{(k-1)\epsilon^{1/d}n^{2/d}\le j\le k\epsilon^{1/d}n^{2/d}}\big\}\Big]\Big)\\ &\quad\le\Big\{\sup_{A\in\mathcal{A}^{(n)}_{\epsilon,N}}\sup_{x\in\mathbb{R}^d}\mathbb{E}_x\Big(\exp\Big[\epsilon^{-\frac{1}{3d}}n^{-\frac{2}{d}}\sharp\big\{A\cap\{S_j\}_{1\le j\le\epsilon^{\frac{1}{d}}n^{\frac{2}{d}}}\big\}\Big]\Big)\Big\}^{\epsilon^{-1/d}n^{(d-2)/d}},\end{aligned} \tag{3.3.51}$$

where $\mathbb{E}_x$ refers to the expectation given that starting point of the random walk with $\mathbb{E}:=\mathbb{E}_0$. Also, we use the Markov property at times $k\epsilon^{1/d}, k=1,\dots,\epsilon^{-1/d}n^{(d-2)/d}$, along with the property that $\mathcal{A}^{(n)}_{\epsilon,N}$ is closed under translations.

(c) Now we consider the expectation from (3.3.51). We use the inequality $e^u\le 1+u+$

$\frac{1}{2}u^2e^u$, along with the Cauchy-Schwarz inequality, to obtain that

$$\mathbb{E}_x\Big(\exp\Big[\epsilon^{-\frac{1}{3d}}n^{-\frac{2}{d}}\sharp\big\{A\cap\{S_j\}_{1\le j\le\epsilon^{\frac{1}{d}}n^{\frac{2}{d}}}\big\}\Big]\Big)\le 1+\epsilon^{-1/3d}n^{-2/d}\mathbb{E}_x\sharp\big\{A\cap\{S_j\}_{1\le j\le\epsilon^{\frac{1}{d}}n^{\frac{2}{d}}}\big\}$$
$$+\frac{1}{2}\epsilon^{4/3d}\sqrt{\frac{1}{(\epsilon^{1/d}n^{2/d})^4}\mathbb{E}_x\big(R_{\epsilon^{1/d}n^{2/d}}\big)^4}\sqrt{\mathbb{E}_x\Big(\exp\big[2\epsilon^{2/3d}\frac{1}{\epsilon^{1/d}n^{2/d}}R_{\epsilon^{1/d}n^{2/d}}\big]\Big)}. \tag{3.3.52}$$

Note that, we over estimate by removing the intersection with A in the last term of (3.3.52). Now, we can see that

$$\frac{1}{(\epsilon^{1/d}n^{2/d})^4}\mathbb{E}_x\big(R_{\epsilon^{1/d}n^{2/d}}\big)^4\le\frac{1}{(\epsilon^{1/d}n^{2/d})^4}\mathbb{E}_x(\epsilon^{1/d}n^{2/d})^4=1, \tag{3.3.53}$$

and

$$\mathbb{E}_x\Big(\exp\big[2\epsilon^{2/3d}\frac{1}{\epsilon^{1/d}n^{2/d}}R_{\epsilon^{1/d}n^{2/d}}\big]\Big)\le\mathbb{E}_x\Big(\exp\big[2\epsilon^{2/3d}\frac{1}{\epsilon^{1/d}n^{2/d}}\epsilon^{1/d}n^{2/d}\big]\Big)$$
$$=\exp[2\epsilon^{2/3d}]. \tag{3.3.54}$$

Combining (3.3.52)–(3.3.54), we get

$$\mathbb{E}_x\Big(\exp\Big[\epsilon^{-\frac{1}{3d}}n^{-\frac{2}{d}}\sharp\big\{A\cap\{S_j\}_{1\le j\le\epsilon^{\frac{1}{d}}n^{\frac{2}{d}}}\big\}\Big]\Big)$$
$$\le 1+\epsilon^{-1/3d}n^{-2/d}\mathbb{E}_x\sharp\big\{A\cap\{S_j\}_{1\le j\le\epsilon^{\frac{1}{d}}n^{\frac{2}{d}}}\big\}+C_1\epsilon^{4/3d}e^{\epsilon^{1/3d}},$$
$$\forall A\subset\mathbb{Z}^d, x\in\mathbb{Z}^d, T\ge 1, 0<\epsilon\le 1. \tag{3.3.55}$$

Finally, the remaining expectation in (3.3.55) can be estimated as follows. Write

$$\begin{aligned}\mathbb{E}_x\sharp\big\{A\cap&\{S_j\}_{1\le j\le\epsilon^{\frac{1}{d}}n^{\frac{2}{d}}}\big\}\\ &=\sum_{z\in\mathbb{Z}^d}\mathbb{E}_x\sharp\big\{A\cap\{S_j\}_{1\le j\le\epsilon^{\frac{1}{d}}n^{\frac{2}{d}}}\cap\Theta_{Nn^{1/d}}(z)\big\}\\ &\le\Big[\sup_{x\in\mathbb{Z}^d}\sup_{z\in\mathbb{Z}^d}\Big\{\mathbb{E}_x\Big\{\sharp\big\{A\cap\{S_j\}_{1\le j\le\epsilon^{\frac{1}{d}}n^{\frac{2}{d}}}\cap\Theta_{Nn^{1/d}}(z)\big\}\Big\}\Big\}\Big]\\ &\qquad\times\mathbb{E}_x\sharp\Big\{z\in\mathbb{Z}^d:\{S_j\}_{1\le j\le\epsilon^{\frac{1}{d}}n^{\frac{2}{d}}}\cap\Theta_{Nn^{1/d}}(z)\neq\emptyset\Big\}\\ &\le\Big[\sup_{x\in\mathbb{Z}^d}\sup_{z\in\mathbb{Z}^d}\Big\{\mathbb{E}_x\Big\{\sharp\big\{A\cap\{S_j\}_{1\le j\le\infty}\cap\Theta_{Nn^{1/d}}(z)\big\}\Big\}\Big\}\Big]\\ &\qquad\times\mathbb{E}_x\sharp\Big\{z\in\mathbb{Z}^d:\{S_j\}_{1\le j\le\epsilon^{\frac{1}{d}}n^{\frac{2}{d}}}\cap\Theta_{Nn^{1/d}}(z)\neq\emptyset\Big\},\end{aligned} \tag{3.3.56}$$

where $\{S_j\}_{1\le j\le\infty}$ is a set of lattice sites visited by an infinite-time random walk. Now, by [AC07], for any $z\in\mathbb{Z}^d$ and $\mathbb{P}_x$ the probability given that random walk start at x, we have

$$\begin{aligned}\mathbb{P}_x\big(\sharp\{A\cap\{S_j\}_{1\le j\le\infty}\cap\Theta_{Nn^{1/d}}(z)\}>t\big)&\le\mathbb{P}_x\big(l_\infty\big(A\cap\Theta_{Nn^{1/d}}(z)\big)>t\big)\\&\le\exp\Big[-\frac{C_2t}{|A\cap\Theta_{Nn^{1/d}}(z)|^{2/d}}\Big],\end{aligned}\tag{3.3.57}$$

where $l_\infty(B)$ is the total time spent by the infinite-time random walk inside a set $B\in\mathbb{Z}^d$. Hence, by the definition of A defined in (3.3.44),

$$\begin{aligned}\mathbb{E}_x\sharp\{A\cap\{S_j\}_{1\le j\le\infty}\cap\Theta_{Nn^{1/d}}(z)\}&\le\int_0^\infty\exp\Big[-\frac{C_2t}{|A\cap\Theta_{Nn^{1/d}}(z)|^{2/d}}\Big]dt\\&=C_2|A\cap\Theta_{Nn^{1/d}}(z)|^{2/d}\\&\le C_2(\epsilon n)^{2/d}.\end{aligned}\tag{3.3.58}$$

Combining (3.3.56) and (3.3.58), we can get from (3.3.55) that,

$$\mathbb{E}_x\sharp\{A\cap\{S_j\}_{1\le j\le\epsilon^{\frac{1}{d}}n^{\frac{2}{d}}}\}\le C_2\epsilon^{2/d}n^{2/d}\mathbb{E}_x\sharp\Big\{z\in\mathbb{Z}^d:\{S_j\}_{1\le j\le\epsilon^{\frac{1}{d}}n^{\frac{2}{d}}}\cap\Theta_{Nn^{1/d}}(z)\neq\emptyset\Big\}.\tag{3.3.59}$$

However,the expectation on the right hand side of (3.3.59) is also bounded above by C_3 uniformly in $x\in\mathbb{Z}^d, N\ge1$ and $0\le\epsilon\le1$. Therefore, by (3.3.55) and (3.3.59)

$$\begin{aligned}&\sup_{x\in\mathbb{Z}^d}\sup_{T\ge1}\mathbb{E}_x\Big(\exp\Big[\epsilon^{-\frac{1}{3d}}n^{-\frac{2}{d}}\sharp\{A\cap\{S_j\}_{1\le j\le\epsilon^{\frac{1}{d}}n^{\frac{2}{d}}}\}\Big]\Big)\\&\qquad\le1+C_2C_3\epsilon^{5/3d}+C_1\epsilon^{4/3d}e^{\epsilon^{1/3d}},\quad\forall0<\epsilon\le1.\end{aligned}\tag{3.3.60}$$

By substituting (3.3.60) into (3.3.51), we then get (3.3.49). This completes the proof of Lemma 3.3.6. □

3.3.5 Proof of Proposition 3.3.1

We complete the proof of Proposition 3.3.1.

Proof. By (3.3.38), (3.3.41), Lemma 3.3.2(a), (c) and Proposition 3.3.5 we have, for n and N large enough, $0\le\epsilon\le1$ and $\delta>0$,

$$\begin{aligned}\mathbb{P}(\tfrac{1}{n}J_n\ge a)\le&\exp\big[-\tfrac12K(\epsilon,\delta)n^{(d-2)/d}\big]\\&+\mathbb{P}\Big(\frac1n\sum_{z\in Z^X_{\epsilon,N}}J^X_{n,\sqrt{N},N,\mathrm{in}}(z)\ge a-2\delta,\mathcal{O}_n\cap\mathcal{C}_{n,\log N,\sqrt{N}}\cap\mathcal{R}^{(2)}_n\Big).\end{aligned}\tag{3.3.61}$$

Now, by Proposition 3.3.4 we have, for any $N \geq 1, 0 < \epsilon \leq 1$ and $\delta > 0$,

$$\begin{aligned}
&\mathbb{P}\Big(\tfrac{1}{n} \sum_{z \in Z^X_{\epsilon,N}} J^X_{n,\sqrt{N},N,\mathrm{in}}(z) \geq a - 2\delta, \mathcal{O}_n \cap \mathcal{C}_{n,\log N,\sqrt{N}} \cap \mathcal{R}^{(2)}_n\Big) \\
&\quad \leq \exp\Big[\gamma_N n^{\frac{d-2}{d}} + O(\log n)\Big] \\
&\qquad \times \mathbb{P}\Big(\tfrac{1}{n} \sum_{z \in Z^X_{\epsilon,N}} J^X_{n,\sqrt{N},N,\mathrm{in}}(z) \geq a - 2\delta, \mathcal{O}_n \cap \mathcal{C}_{n,\log N,\sqrt{N}} \cap \mathcal{R}^{(2)}_n \cap \mathcal{D}\Big),
\end{aligned} \tag{3.3.62}$$

with $\mathcal{D}$ the disjointness property stated in Proposition 3.3.4(a). However, by this disjointness property we have

$$\tfrac{1}{n}\mathfrak{J}_n \geq \tfrac{1}{n} \sum_{z \in Z^X_{\epsilon,N}} J^X_{n,\sqrt{N},N,\mathrm{in}}(z), \tag{3.3.63}$$

where $\mathfrak{J}_n$ is the number of intersection points wrapped around $\Lambda_{2^{4\kappa/\epsilon}N}$, the torus of size $2^{4\kappa/\epsilon}N$, i.e., $\mathfrak{J}_n = \sharp\big\{\{\mathfrak{S}^1_j\}_{1\leq j\leq n} \cap \{\mathfrak{S}^2_j\}_{1\leq j\leq n}\big\}$ where $\mathfrak{S}^i_j$ is the position of the random walk when wrapped around $\Lambda_{2^{4\kappa/\epsilon}Nn^{1/d}}$. Note that we use that fact that $\sharp\{Z^X_{\epsilon,N}\} \leq 4\kappa/\epsilon$ on $\mathcal{R}^{(2)}_n$. Combining $(3.3.61)-(3.3.63)$ we obtain that, for n, N large enough, $0 < \epsilon \leq 1$ and $\delta > 0$,

$$\mathbb{P}(\tfrac{1}{n}J_n \geq a) \leq e^{-\frac{1}{2}K(\epsilon,\delta)n^{(d-2)/d}} + e^{\gamma_N n^{\frac{d-2}{d}} + O(\log n)}\mathbb{P}\Big(\tfrac{1}{n}\mathfrak{J}_n \geq a - 2\delta\Big). \tag{3.3.64}$$

We then use Proposition 3.1.1 to obtain that, for N large enough, $0 < \epsilon \leq 1$ and $\delta > 0$,

$$\limsup_{n\to\infty} \frac{1}{n^{\frac{d-2}{d}}} \log \mathbb{P}\big(\tfrac{1}{n}J_n \geq a\big) \leq \max\Big\{ -\tfrac{1}{2}K(\epsilon,\delta), \gamma_N - \hat{L}^\kappa_{2^{4\kappa/\epsilon}N}(a - 2\delta)\Big\}. \tag{3.3.65}$$

Next, we let $N \to \infty$ and use the facts that $\gamma_N \to 0$ and note that

$$\lim_{N\to\infty} \hat{L}^\kappa_{2^{4\kappa/\epsilon}N}(a - 2\delta) = L^\kappa(a - 2\delta),$$

by Proposition 1.3.10. We then obtain that, for any $0 < \epsilon \leq 1$ and $\delta > 0$,

$$\limsup_{n\to\infty} \frac{1}{n^{\frac{d-2}{d}}} \log \mathbb{P}(\tfrac{1}{n}J_n \geq a) \leq \max\Big\{ -\tfrac{1}{2}K(\epsilon,\delta), -L^\kappa(a - 2\delta)\Big\}. \tag{3.3.66}$$

Next, let $\epsilon \downarrow 0$ which gives $K(\epsilon,\delta) \to \infty$, to obtain that, for any $\delta > 0$,

$$\limsup_{n\to\infty} \frac{1}{n^{\frac{d-2}{d}}} \log \mathbb{P}(\tfrac{1}{n}J_n \geq a) \leq -L^\kappa(a - 2\delta). \tag{3.3.67}$$

Finally we need to show that $L^\kappa(a-2\delta)$ converges to our required rate function, $L^\kappa(a)$, in Theorem 1.3.7. We refer to the results by [BBH04] to show this, since we can write

L^κ in term of $\hat{I}_d^\kappa$ defined in (1.3.32). First of all, by the scaling relations in (1.3.34) and (1.3.35) we have

$$L^\kappa(a-2\delta) = \big(1-\tfrac{2\delta}{a}\big)^{\frac{d-2}{d}} L^{\kappa/(1-2\delta)}(a). \tag{3.3.68}$$

Now, we are in position to transform L^κ to another rate function $\hat{\Theta}_d$ by the relation described in (1.3.23) where $\hat{\Theta}_d$ is also described in (1.3.23), except we replace κ_a by κ. The benefit of transforming the rate function is that $\hat{\Theta}_d$ is continuous at a. Hence, we can pass the limit $\delta \downarrow 0$ and therefore

$$L^\kappa(a-2\delta) \xrightarrow{\delta\downarrow 0} L^\kappa(a).$$

This completes the proof of Proposition 3.3.1 and hence Theorem 1.3.7. $\square$

Chapter 4

Summary and open problem

In this chapter, we make a summary of the results in Chapter 2 and Chapter 3. We also point out an open problem in the range of random walk and make a comment on the problem.

The large deviation behaviour problems on the range of a random walk and on the intersections of the independent ranges have been solved, respectively, in Chapter 2 and Chapter 3. We have considered the problem on the range in the downward direction, and the problem on the intersections in the upward direction. In both cases, the speed of the large deviation are both $n^{(d-2)/d}$ and the rate functions are explicitly given in Theorem 1.2.10 and Theorem 1.3.7. For the result in Theorem 1.2.10, the main technique for the proof is to project the random walk on a time-dependence torus in order to get a good control for the range. The size of the torus will later increase to infinity. While for the result in Theorem 1.3.7, the main technique is to reflect the random walks in order to move the main contribution of the intersections of the ranges inside a large time-dependence box. Then, we can apply the result for the intersections of the ranges on the torus to get the large deviation result.

We try to extend our result in the general case. We now concentrate in the downward direction. Rather than getting the large deviation principle on the range, we may consider the large deviation problem of the type:

$$\mathbb{P}\Big(\sum_{x \in \{S_1, \ldots, S_n\}} f(xn^{-1/d}) \approx n \Big).$$

Note that, by taking a constant function,

$$f(x) = \frac{1}{\kappa}, \qquad \text{for all } x,$$

this is exactly the same problem as in Theorem 1.2.10 which has been studied in Chapter 2. Therefore, this suggests the conjecture:

Conjecture 4.1. *For $d \geq 3$, and a function f bounded away from 0 and ∞,*

$$\lim_{n\to\infty} \frac{1}{n^{\frac{d-2}{d}}} \log \mathbb{P}\Big(\sum_{x\in\{S_1,\ldots,S_n\}} f(xn^{-1/d}) \leq n \Big) = -I^{\kappa}(f),$$

where $I^{\kappa}(f)$ is an explicitly given rate function.

This conjecture agrees with the conjecture made by Chen(Equation (7.6.2), [Che10]).

However, it seems that both techniques from Chapter 2 and 3 cannot be applied to solve this conjecture:

- By projecting the random walk to $\Lambda_{Nn^{1/d}}$, for the range we would get $\mathcal{R}_n \leq R_n$ which gives the upper bound in probability for the event. However, this argument does not imply the same outcome here. If we consider a point $x \in \Lambda_{Nn^{1/d}}$ (with the assumption that x is on a scale grid) then x can be any point in the set $\{x + zNn^{1/d} : z \in \mathbb{Z}^d\}$ at which each point in the set give the different values on the function f. Also, the existence of x on the torus may represent the multiple points for $\mathbb{Z}^d$. Hence, by projecting the random walk on a torus, we can not conclude that the sum will increase or decrease.
- By reflection technique described in Chapter 3, the problem of the point x represents multiple points will disappear. However, this technique still not good enough to solve the conjecture by the lack of monotonicity of the function f, which is not the case for the intersections of the ranges.

Nevertheless, if we add a condition that the function f is radially decreasing, then we can apply the reflection technique to solve the conjecture. By making reflection, according to the location of the popular boxes, we end up with the reflected random walk which stay closer to the origin than the original walk. This gives the lower bound in probability for the sum. The projection technique, however, still cannot be applied here since the point x on the torus may still represent the multiple points on $\mathbb{Z}^d$.

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